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سایت آموزش مهندسی مکانیک ایران

## 2.1 Rectilinear (Straight Line) Motion

The position of a particle $P$ on a straight line can be specified by its distance from the origin of a reference frame along any one of the reference axes. Without loss of generality we can use the $x$ axis of a Cartesian reference frame as the line along which the motion of the particle takes place. Let the distance of $P$ from the origin of the frame be denoted by $s$. Then the position vector of $P$ can be written as:

$$\mathbf{r} = s\mathbf{i} \tag{2.1}$$

where $\mathbf{i}$ is a unit vector along the $x$ axis.

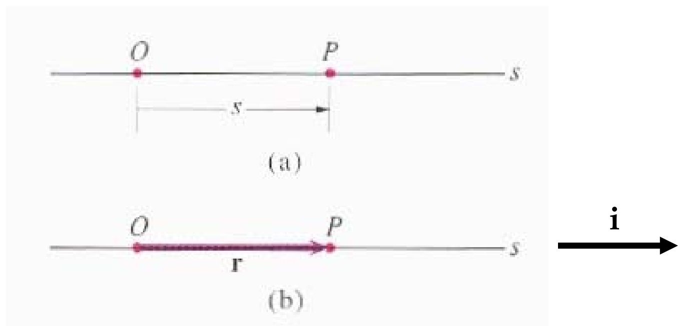


**Figure 1**. Position of a particle $P$ in rectilinear motion

Now the velocity $\mathbf{v}$ of $P$ can be written as:

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = \frac{ds}{dt}\mathbf{i} + s\frac{d\mathbf{i}}{dt} = \frac{ds}{dt}\mathbf{i} = \dot{s}\mathbf{i} = v\mathbf{i} \tag{2.2}$$

The acceleration $\mathbf{a}$ of $P$ can now be written as:

$$\mathbf{a} = \frac{d\mathbf{v}}{dt} = \frac{dv}{dt}\mathbf{i} + v\frac{d\mathbf{i}}{dt} = \frac{dv}{dt}\mathbf{i} = \dot{v}\mathbf{i} = a\mathbf{i} \tag{2.3}$$

Note:

a) the chain rule is used in differentiating the expressions for **r** and **v**
b) the derivative $\frac{d\mathbf{i}}{dt} = 0$ because neither the magnitude nor the direction of **i** changes with time
c) the notation $\dot{s}$ is used to denote $\frac{ds}{dt}$ and $\dot{v}$ is used to denote $\frac{dv}{dt}$
d) the velocity **v** and acceleration a of the particle are still vectors but the magnitudes of these vectors, $v$ and $a$, are scalars
e) the magnitude of the velocity, $v$, is often called the speed

Given that the displacement, velocity, and acceleration of the particle are all associated with the same unit vector **i** in rectilinear motion, it makes sense to write everything in terms of the magnitudes of these quantities only. In other words:

$$v = \dot{s} = \frac{ds}{dt} \tag{2.4}$$

and

$$a = \dot{v} = \frac{dv}{dt} \tag{2.5}$$

From these two expressions two additional expressions can be written for $a$:

$$a = \frac{dv}{dt} = \frac{d}{dt}\left(\frac{ds}{dt}\right) = \frac{d^2 s}{dt^2} = \ddot{s} \tag{2.6}$$

and

$$a = \frac{dv}{dt} = \frac{dv}{ds}\frac{ds}{dt} = v\frac{dv}{ds} \tag{2.7}$$

Note:

a) the notation $\ddot{s}$ is used to denote $\dfrac{d^2 s}{dt^2}$
b) the expression in Eq. (2.7) introduces only a change of differentiation variable and does not introduce a new definition of acceleration

## 2.2 Integrals of Rectilinear Motion

The expressions above show how the velocity and acceleration of a particle in rectilinear motion can be obtained if the displacement of the particle is known as a function of time (or using Eq. (2.7) if the particle velocity is known as a function of its displacement). Often it is the acceleration of the particle that is known as a function of time or displacement and it is desired to compute the velocity and/or displacement of the particle as a function of time and/or displacement. To do that the equations above have to be written in a different form.

From Eq. (2.5) we can write:

$$dv = a\,dt \tag{2.8}$$

and consequently:

$$\int_{v_0}^{v} dv = \int_{t_0}^{t} a(t)\,dt \tag{2.9}$$

where $v_0$ is the velocity of the particle at $t_0$ and $v$ is the velocity of the particle at time $t$.

When the expression above is evaluated we obtain:

$$v(t) - v_0 = \int_{t_0}^{t} a(t)\,dt \tag{2.10}$$

or

$$v(t) = v_0 + \int_{t_0}^{t} a(t)\,dt \tag{2.11}$$

Note:

a) the notation $a(t)$ indicates that the acceleration $a$ is an explicit function of time $t$, which is necessary for performing the integration
b) the evaluation of the integral on the right side results in a velocity $v$ that is an explicit function of time, hence the notation $v(t)$
c) the term $v_0$ is a constant and denotes the velocity of the particle at time $t_0$ (or $v_0 = v(t_0)$)

Similarly from Eq. (2.4) we can write:

$$ds = v\,dt \tag{2.12}$$

and consequently

$$\int_{s_0}^{s} ds = \int_{v_0}^{v} v(t)\,dt \tag{2.13}$$

This results in

$$s(t) = s_0 + \int_{v_0}^{v} v(t)\,dt \tag{2.14}$$

Click here for an example on the use of the expressions above

## 2.3 Specialization to constant acceleration

A commonly encountered situation is the case of constant acceleration (e.g. gravity or a particle accelerating under action of a constant force). If the acceleration of a particle in rectilinear motion is constant its velocity and displacement as a function of time can be readily obtained from Eqs. (2.11) and (2.14). Specifically Eq. (2.11) results in:

$$v(t) = v_0 + a_0 (t - t_0) \tag{2.15}$$

and Eq (2.14) yields:

$$s(t) = s_0 + v_0 (t - t_0) + \frac{1}{2} a_0 (t - t_0)^2 \tag{2.16}$$

where $a_0$ is the constant acceleration of the particle.

For $t_0 = 0$ the expressions above reduce to the familiar relations:

$$\begin{aligned} v(t) &= v_0 + a_0 t \\ s(t) &= s_0 + v_0 t + \frac{1}{2} a_0 t^2 \end{aligned} \tag{2.17}$$

For the constant acceleration case an additional result can be obtained from Eq. (2.7). Rewriting equation in a slightly different form we obtain:

$$a_0 \, ds = v \, dv \tag{2.18}$$

from which we can write:

$$\int_{s_0}^{s} a_0 \, ds = \int_{v_0}^{v} v \, dv \tag{2.19}$$

Evaluation of this expression results in:

$$v(s) = \left[ v_0^2 + 2a_0 (s - s_0) \right]^{\frac{1}{2}} \tag{2.20}$$

Note:

a) The expressions in Eqs. (2.17) and (2.20) may look familiar but care must be taken in using them to ensure that the acceleration is constant.

Click here for an example on the use of the expressions above
Click here for a second example on the use of the expressions above

## 2.4 Other Forms of First Integrals of Motion

Expressions of the type given by Eqs. (2.11) and (2.20) are known as *first integrals of motion* because they are obtained by integrating the acceleration once. Other forms of first integrals can also be obtained depending on how the acceleration is expressed. The expression given by Eq. **Error! Reference source not found.** is obtained if the acceleration is expressed as a function of time. However it is also possible that the acceleration of the particle is expressed in terms of its displacement and its velocity. In such instances slightly different techniques are required.

If the acceleration is expressed as a function of the particle's velocity (this occurs when, for example, a particle is traveling against air or fluid resistance) we can write:

$$\frac{dv}{dt} = a(v) \tag{2.21}$$

which can also be written as:

$$\frac{dv}{a(v)} = dt \tag{2.22}$$

In integral form this becomes:

$$\int_{v_0}^{v} \frac{dv}{a(v)} = \int_{t_0}^{t} dt = t - t_0 \tag{2.23}$$

where $v_0$ and $t_0$ have the same meanings as before.

If the acceleration is expressed as a function of the particle's displacement (this occurs when a particle is displaced against the resistance of a spring or a Newtonian gravitational field) we can write from Eq. (2.7):

$$v\,dv = a(s)\,ds \tag{2.24}$$

In integral form this becomes:

$$\int_{s_0}^{s} a(s)\,ds = \int_{v_0}^{v} v\,dv = \frac{1}{2}\left(v^2 - v_0^2\right) \tag{2.25}$$

which is a more general form of Eq. (2.20).

Eq. (2.24) can be used to obtain another first integral of motion if the acceleration is expressed as a function of velocity. In this case Eq. **Error! Reference source not found.** can be rewritten as:

$$v\,dv = a(v)\,ds \tag{2.26}$$

In integral form this becomes:

$$\int_{v_0}^{v} \frac{v\,dv}{a(v)} = \int_{s_0}^{s} ds = s - s_0 \tag{2.27}$$

Click here for an example on velocity-dependent acceleration
Click here for an example on displacement-dependent acceleration