3. Curvilinear Motion

As seen in Module 2 rectilinear motion can be described by the scalar parameters s, v, and a. The most general motion of a particle, however, is *curvilinear* in three-dimensional Cartesian space. More specifically this means that the path of the particle relative to some reference frame is a three-dimensional curve. The specific mathematical expression describing the curve depends on the reference frame and the coordinate system used. Most generally the curvilinear motion of a particle in three-dimensional space is described by vectors. Vector expressions for the position of a particle using different coordinate systems were written in Module 1 (Eqs. (1.2)-(1.4)). Rewriting Eq. (1.2):

$$\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k} \tag{3.1}$$

Using the definition of velocity and acceleration (Eqs. (1.5) and (1.6)) and vector differentiation rules, vector expressions for the velocity and acceleration of the particle can be obtained:

$$\begin{aligned}\mathbf{v} &= \frac{d\mathbf{r}}{dt} = \frac{dx}{dt}\mathbf{i} + \frac{dy}{dt}\mathbf{j} + \frac{dz}{dt}\mathbf{k} + x\frac{d\mathbf{i}}{dt} + y\frac{d\mathbf{j}}{dt} + z\frac{d\mathbf{k}}{dt} \\ &= \frac{dx}{dt}\mathbf{i} + \frac{dy}{dt}\mathbf{j} + \frac{dz}{dt}\mathbf{k} \\ &= \dot{x}\mathbf{i} + \dot{y}\mathbf{j} + \dot{z}\mathbf{k} \\ &= v_x\mathbf{i} + v_y\mathbf{j} + v_z\mathbf{k}\end{aligned} \tag{3.2}$$

and

$$
\begin{aligned}
\mathbf{a} = \frac{d\mathbf{v}}{dt} &= \frac{dv_x}{dt}\mathbf{i} + \frac{dv_y}{dt}\mathbf{j} + \frac{dv_z}{dt}\mathbf{k} + v_x\frac{d\mathbf{i}}{dt} + v_y\frac{d\mathbf{j}}{dt} + v_z\frac{d\mathbf{k}}{dt} \\
&= \frac{dv_x}{dt}\mathbf{i} + \frac{dv_y}{dt}\mathbf{j} + \frac{dv_z}{dt}\mathbf{k} \\
&= \dot{v}_x\mathbf{i} + \dot{v}_y\mathbf{j} + \dot{v}_z\mathbf{k} = \ddot{x}\mathbf{i} + \ddot{y}\mathbf{j} + \ddot{z}\mathbf{k} \\
&= a_x\mathbf{i} + a_y\mathbf{j} + a_z\mathbf{k}
\end{aligned}
\qquad (3.3)
$$

These expressions can be written in scalar form as:

$$
v_x = \frac{dx}{dt}; \quad v_y = \frac{dy}{dt}; \quad v_z = \frac{dz}{dt} \qquad (3.4)
$$

and

$$
a_x = \frac{dv_x}{dt}; \quad a_y = \frac{dv_y}{dt}; \quad a_z = \frac{dv_z}{dt}; \qquad (3.5)
$$

Note:

a) the chain rule is used in differentiating the expressions for **r** and **v**

b) the derivatives $\frac{d\mathbf{i}}{dt} = \frac{d\mathbf{j}}{dt} = \frac{d\mathbf{k}}{dt} = \mathbf{0}$ because neither the magnitude nor the direction of unit vectors associated with Cartesian coordinate systems change with time (this is not true for other coordinate systems such as cylindrical or spherical)

c) the scalar equations (Eqs. (3.4) and (3.5)) are identical in form to Eqs. (2.4) and (2.5) for rectilinear motion; consequently the methods developed for rectilinear motion can be applied to each coordinate direction independently.

3.1 Motion of a Projectile

The *projectile problem* is a classical example independent accelerations in two different directions. In this class of problems at $t = 0$ a particle is launched from the surface of the earth with a speed of v_0 at an angle of θ_0 above the horizontal (see Fig. 1).

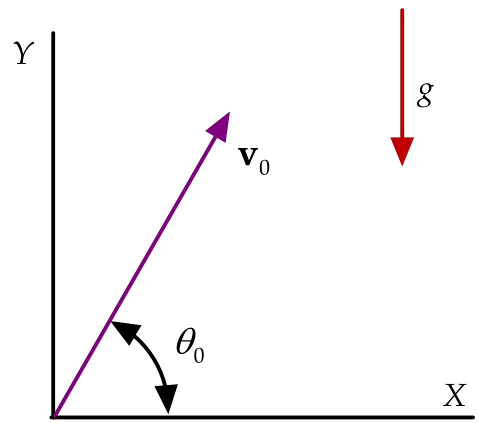

Figure 1. Initial conditions for the projectile problem

For simplicity we assume that all motion occurs in vacuum (i.e. no air resistance) and that the particle is subject only to gravitational acceleration. We also assume that all motion occurs in a single plane. Under these assumptions we need only two Cartesian coordinates, x and y, to develop expressions for the motion of the projectile. Let us assume that the projectile is launched from the origin of a Cartesian reference frame at $t = 0$. From the statement of the problem and the assumptions made we can write the following expressions:

$$\begin{aligned} &x(0) = 0; \quad y(0) = 0; \\ &v_x(0) = v_0 \cos\theta_0; \quad v_y(0) = v_0 \sin\theta_0; \\ &a_x = 0; \quad a_y = -g \end{aligned} \tag{3.6}$$

Note:

a) the first pair of expressions state that the projectile is launched from the origin of the Cartesian reference frame at $t = 0$
b) the second pair of expressions specify the velocity of the projectile in the X and Y directions at $t = 0$
c) expressions that define the position and velocity of the particle at $t = 0$ are known as *initial conditions*
d) the third pair of expressions define the acceleration of the projectile in the X and Y directions (for all time)
e) the symbol g is used to denote the gravitational acceleration constant (32.2 ft/s^2 or 9.8 m/s^2)
f) the acceleration of the particle in the y direction is stated as a negative quantity because it is in the negative Y direction relative to the reference frame chosen

It is evident from these expressions that the projectile has constant acceleration (of different magnitudes) in the X and Y directions. Because both accelerations are constant we can apply the results obtained in Module 2 for constant acceleration in both directions. Applying Eqs. (2.15) and (2.16) to the motion in the x direction we obtain for $v_x(t)$ and $x(t)$ (using $t_0 = 0;\, a_0 = 0;\, v(t_0) = v_0 \cos\theta_0$):

$$v_x(t) = v_0 \cos\theta_0 \tag{3.7}$$

$$x(t) = (v_0 \cos\theta_0)\,t = v_0 t \cos\theta_0 \tag{3.8}$$

Similarly applying Eqs. (2.15) and (2.16) to the motion in the Y direction we obtain for $v_y(t)$ and $y(t)$ (using $t_0 = 0;\; a_0 = -g;\; v(t_0) = v_0 \sin\theta_0$):

$$v_y(t) = v_0 \sin\theta_0 - gt \tag{3.9}$$

$$y(t) = (v_0 \sin\theta_0)t - \frac{1}{2}gt^2 \tag{3.10}$$

Note:

a) the motions of the particle in the two directions are completely independent of each other
b) both motions were derived from expressions for rectilinear constant acceleration motion
c) the motion of the particle in the X direction varies linearly with t while the motion in the Y direction varies quadratically with t

The curve or trajectory that the projectile describes relative to the reference frame in which its motion was defined can be obtained by eliminating t from Eqs. (3.8) and (3.10). This results in the equation of a *parabola* given by:

$$y = (\tan\theta_0)x - \frac{g}{2v_0^2 \cos^2\theta_0}x^2 \tag{3.11}$$

Click to see an animation of the trajectory of a projectile

Click here for a projectile example
Click here for an example of nonprojectile curvilinear motion