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4 Angular Motion

Although Cartesian coordinates may be sufficient and/or appropriate for solving certain classes of problems involving curvilinear motion, other coordinate systems such as cylindrical or spherical may be useful for several other classes of problems. An important ingredient of such analysis is the concept of *angular motion*.

4.1 Definition of Angular Velocity and Acceleration

The angular position of a line L in an arbitrary plane relative to a reference line L_0 can be specified by an θ (see Fig. 1).

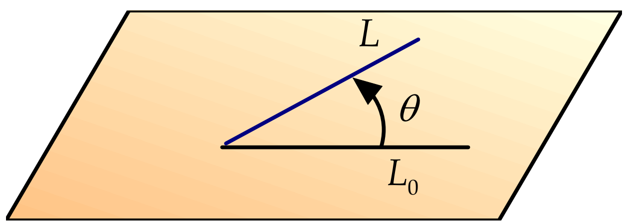

Figure 1. Angular position of a line relative to a reference line

Click to see an animation rotation of line *L*

Then the *angular velocity* ω of L relative to L_0 is defined by:

$$\omega = \frac{d\theta}{dt} = \dot{\theta} \tag{4.1}$$

and the *angular acceleration* α of L relative to L_0 is defined by:

$$\alpha = \frac{d\omega}{dt} = \frac{d^2\theta}{dt^2} = \ddot{\theta} \tag{4.2}$$

The units of angular velocity and acceleration must always be rad/s and rad/s² in calculations.

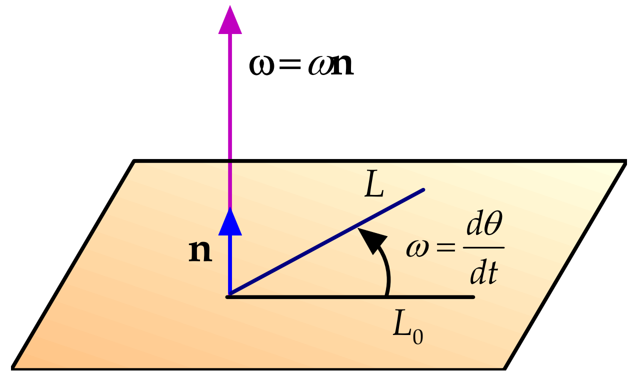

Figure 2. Angular velocity of a line represented as a vector normal to plane of rotation

Click to see an animated figure describing the angular velocity vector

The definitions given above are the simplest descriptions of angular velocity and acceleration. A more general description of angular velocity and acceleration takes into account the direction of the plane in which θ is measured. If we denote by **n** the unit vector normal to the plane in which θ is measured (i.e. the plane of L and L_0) (see Fig. 2) in the angular velocity *vector* **ω** is defined as:

$$\boldsymbol{\omega} = \omega \mathbf{n} = \frac{d\theta}{dt}\mathbf{n} = \dot{\theta}\mathbf{n} \tag{4.3}$$

Similarly the angular acceleration vector **α** is defined as:

$$\boldsymbol{\alpha} = \alpha \mathbf{n} = \frac{d^2\theta}{dt^2}\mathbf{n} = \ddot{\theta}\mathbf{n} \tag{4.4}$$

Note:

a) like all vectors the vector **n** (i.e. the orientation of the plane in which θ is measured) has to be described with respect to a reference frame
b) a plane can be defined uniquely relative to a reference frame by a unit vector
c) by convention the relationship between the direction of **n**, ω, or α and the direction of angular motion in the plane of L and L_0 is defined by the right-hand rule; when the thumb of the right hand points in the direction of **n**, ω, or α the direction of the curved fingers shows the direction of angular motion (see Fig.3

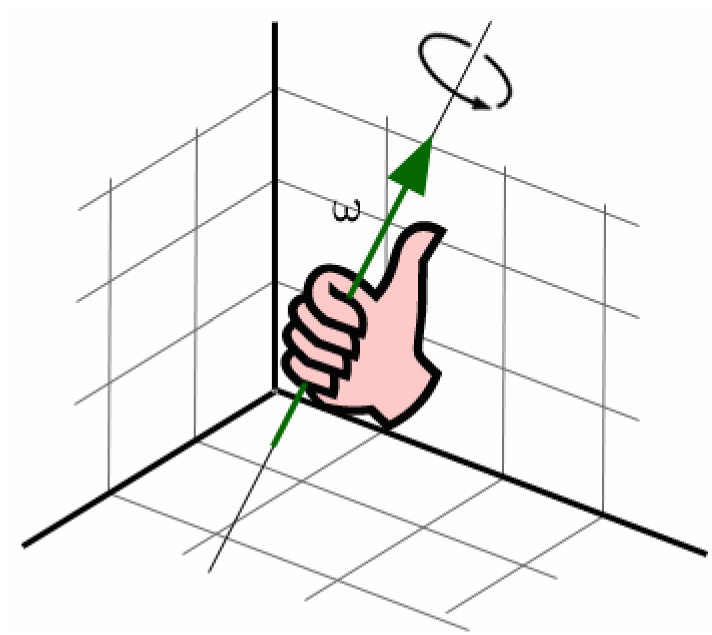

Figure 3 Right-hand rule for defining the direction of an angular velocity vector

4.2 Rate of Change of a Rotating Unit Vector

The directions the unit vectors **i**, **j**, **k** which are attached to the Cartesian reference frame are always fixed. As a result their derivatives with respect to time vanish because neither their magnitudes nor their directions change with time. However unit vectors associated with other coordinate systems (e.g. cylindrical or spherical) change direction with time (the magnitude of a unit vector never changes from unity). Consequently their derivatives with respect to time are nonzero.

Consider a unit vector **e** which rotates instantaneously around an axis defined by a unit vector **n** (see Fig. 4). Let the projection of **e** on the plane perpendicular to **n** be denoted by $\mathbf{e}_\perp$. Then as **e** rotates

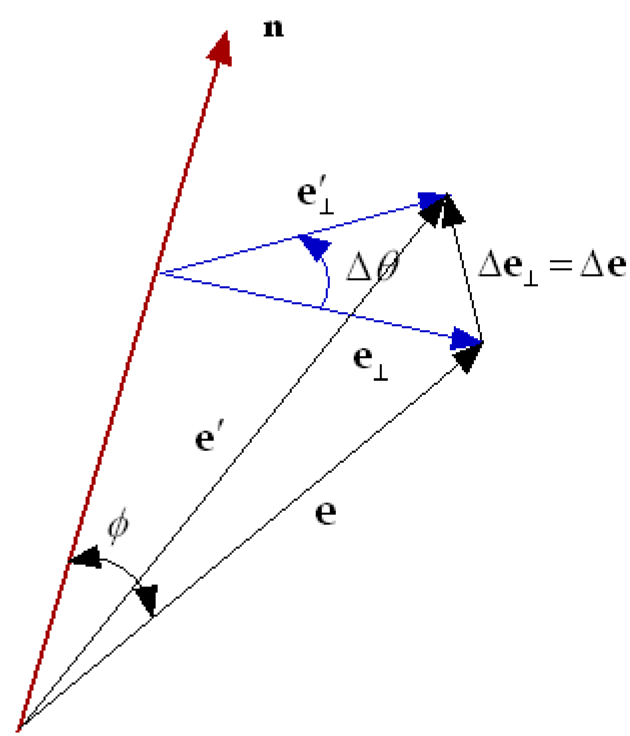

Figure 4 Rotation of a unit vector about an axis **n**

around $\mathbf{n}$ to $\mathbf{e}'$ its projection $\mathbf{e}_\perp$ rotates to $\mathbf{e}'_\perp$. It is clear that the change in $\mathbf{e}$, $\mathbf{\Delta e}$, is given by:

$$\mathbf{\Delta e} = \mathbf{e}' - \mathbf{e} = \mathbf{e}'_\perp - \mathbf{e}_\perp \tag{4.5}$$

From the figure it is evident that $\mathbf{e}_\perp$, $\mathbf{e}'_\perp$, and $\mathbf{\Delta e}$ form an isosceles triangle with a vertex angle of $\Delta\theta$ since $\mathbf{e}_\perp$ and $\mathbf{e}'_\perp$ have the same magnitude. Then the magnitude of $\mathbf{\Delta e}$ can be computed as:

$$|\mathbf{\Delta e}| = 2|\mathbf{e}_\perp|\sin\frac{\Delta\theta}{2} = 2|\mathbf{e}'_\perp|\sin\frac{\Delta\theta}{2} \tag{4.6}$$

However the magnitude of $\mathbf{e}_\perp$ or $\mathbf{e}'_\perp$ can be readily computed from:

$$|\mathbf{e}_\perp| = |\mathbf{e}'_\perp| = |\mathbf{e}|\sin\phi = \sin\phi \tag{4.7}$$

where ϕ is the (constant) angle between $\mathbf{e}$ and $\mathbf{n}$. Note also that $|\mathbf{e}| = |\mathbf{e}'| = 1$ by definition. Substituting this result in Eq. (4.6) we obtain:

$$|\mathbf{\Delta e}| = 2\sin\frac{\Delta\theta}{2}\sin\phi \tag{4.8}$$

Since we are assuming that the change in $\mathbf{e}$ occurs instantaneously we have to assume that $\Delta\theta$ is small and that it takes place over a small interval of time Δt. From basic series expansion theory we know that for small $\Delta\theta$,

$$\sin\frac{\Delta\theta}{2} \approx \frac{\Delta\theta}{2}$$

Thus Eq. (4.8) reduces to:

$$|\mathbf{\Delta e}| \approx \Delta\theta\sin\phi \tag{4.9}$$

Now if we want to find the rate of change of the magnitude of **e** with respect to time we write:

$$\frac{|\Delta \mathbf{e}|}{\Delta t} = \left|\frac{\Delta \mathbf{e}}{\Delta t}\right| \approx \frac{\Delta\theta}{\Delta t}\sin\phi \tag{4.10}$$

Finally taking the limit of this expression as $\Delta t \to 0$ we obtain:

$$\left|\frac{d\mathbf{e}}{dt}\right| = \frac{d\theta}{dt}\sin\phi = \omega\sin\phi \tag{4.11}$$

Eq. (4.11) defines the *magnitude of the rate of change* of **e** (not the rate of change of magnitude of **e** which is zero). The direction of the rate of change of **e** can be deduced from Fig. 4. It is evident that $\Delta\theta \to 0$ as $\Delta t \to 0$. Consequently in the limit the vector $\Delta\mathbf{e}$ becomes perpendicular to $\mathbf{e}_\perp$. Since $\Delta\mathbf{e}$ lies in the plane that is perpendicular to **n** it is also perpendicular to **n** (all vectors in that plane are perpendicular to **n**). Therefore, in the limit, $\Delta\mathbf{e}$ is also perpendicular to the plane made up of **n** and **e**. Since $\frac{d\mathbf{e}}{dt}$ has the same direction as $\Delta\mathbf{e}$ in the limit, the vector $\frac{d\mathbf{e}}{dt}$ is perpendicular to **n** and the plane of **e** and **n**. This directional relationship amongst **e**, **n**, and $\frac{d\mathbf{e}}{dt}$ and the magnitude relationship expressed by Eq. (4.11) can be expressed by the important vector relationship:

$$\frac{d\mathbf{e}}{dt} = \dot{\mathbf{e}} = \omega\mathbf{n}\times\mathbf{e} = \boldsymbol{\omega}\times\mathbf{e} \tag{4.12}$$

Note:

a) Eq. (4.12) expresses both the magnitude relationship expressed by Eq. (4.11) and the directional relationship described by the discussion above
b) Both $\boldsymbol{\omega}$ and **e** must be expressed in the same reference frame to apply the rules of the cross product operation

c) Eq. (4.12) expresses an instantaneous relationship between $\boldsymbol{\omega}$ and **e**; it defines the instantaneous rate of change of **e** with respect to time in terms of its instantaneous rate of rotation around an instantaneous axis of rotation

4.3 Planar Angular Kinematics

The definitions for angular velocity and acceleration given by Eqs. (4.1) and (4.2) are identical to the linear expressions given by Eqs. (2.4) and (2.5). The only difference between them is that the differentiated variables in Eqs. (4.1) and (4.2) are angular displacements and velocities as opposed to the linear displacements and velocities in Eqs. (2.4) and (2.5). Because of this one-to-one correspondence the results obtained for rectilinear motion in Section 2 can be directly applied to planar (but not three-dimensional) angular motion.

Thus the angular analog of Eq. (2.7) is:

$$\alpha = \frac{d\omega}{dt} = \frac{d\omega}{d\theta}\frac{d\theta}{dt} = \omega\frac{d\omega}{d\theta} \tag{4.13}$$

The angular analogs of Eqs. (2.11) and (2.14) are:

$$\omega(t) = \omega_0 + \int_{t_0}^{t} \alpha(t)\,dt \tag{4.14}$$

$$\theta(t) = \theta_0 + \int_{t_0}^{t} \omega(t)\,dt \tag{4.15}$$

Similarly for constant angular acceleration α_0 we can write from Eqs. (2.17) and (2.20)

$$\begin{aligned} \omega(t) &= \omega_0 + \alpha_0 t \\ \theta(t) &= \theta_0 + \omega_0 t + \frac{1}{2}\alpha_0 t^2 \end{aligned} \tag{4.16}$$

$$\omega(\theta) = \left[\omega_0^2 + 2\alpha_0\left(\theta - \theta_0\right)\right]^{\frac{1}{2}} \tag{4.17}$$

In cases of displacement or velocity dependent angular motion, Eqs. (2.23), (2.25), and (2.27) can be adapted to express angular analogs.

Click here for an example on planar angular kinematics