

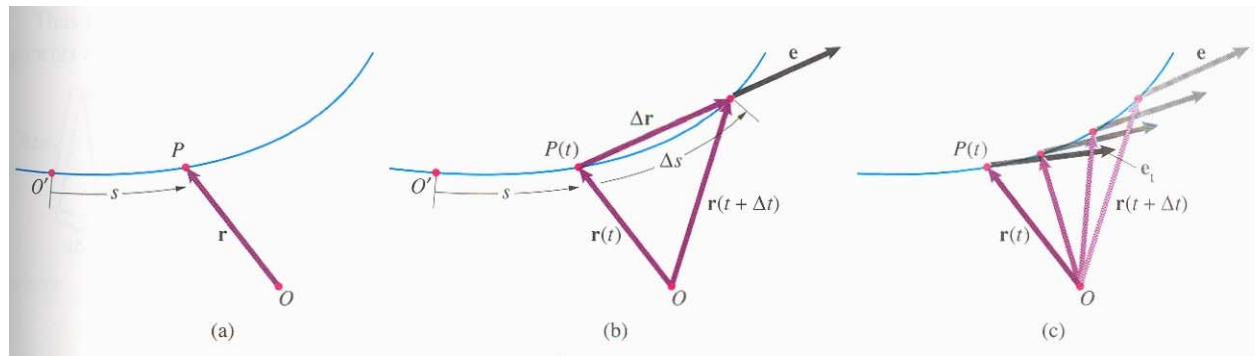
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سایت آموزش مهندسی مکانیک ایران

## 5. General Plane Motion Described in Path Coordinates

It is sometimes convenient to describe the motion of a particle in terms of a coordinate measured along a general path (similar to rectilinear motion except that the displacement of the particle is measured along a curve). For the purposes of this course we will restrict ourselves to motion along a *planar curve*.

### 5.1. Velocity of a particle in path coordinates



**Figure 1.** Motion of a particle along a path

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Consider a particle  $P$  moving along a planar curve. Let the position of the particle at any time  $t$  be specified by a vector  $\mathbf{r}(t)$  defined relative to a reference point  $O$  (see Fig. 1(a)). Let the *path coordinate*  $s(t)$  measure the displacement of the particle from a reference point  $O'$  on the curve (i.e. we assume that the particle was at  $O'$  at time  $t_0$ ). The velocity  $\mathbf{v}$  of the particle relative to  $O$  is defined by Eq. (1.5):

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = \lim_{\Delta t \rightarrow 0} \frac{\mathbf{r}(t + \Delta t) - \mathbf{r}(t)}{\Delta t} = \lim_{\Delta t \rightarrow 0} \frac{\Delta \mathbf{r}}{\Delta t} \quad (5.1)$$

where the vector  $\Delta \mathbf{r}$  is as shown in Fig. 1(b).

During the same interval  $\Delta t$  the particle travels a distance of  $\Delta s$  as measured along the curve. In terms of  $s$  the expression above can be written as:

$$\mathbf{v} = \lim_{\Delta t \rightarrow 0} \frac{\Delta s}{\Delta t} \mathbf{e} \quad (5.2)$$

where  $\mathbf{e}$  is a unit vector collinear with and in the direction of  $\Delta \mathbf{r}$  (see Fig.1(c)). As  $\Delta t \rightarrow 0$   $\mathbf{e}$  becomes a unit vector tangent to the path of  $P$  at its position at time  $t$ . We denote the unit vector at that position by  $\mathbf{e}_t$  which is known as the *tangent unit vector*. Then Eq. (5.2) becomes:

$$\mathbf{v} = \frac{ds}{dt} \mathbf{e}_t = s\dot{\mathbf{e}}_t = v\mathbf{e}_t \quad (5.3)$$

#### Note

- a) this expression is very similar to the expression given by Eq. (2.2) but in this case the unit vector  $\mathbf{e}_t$  is not constant in direction relative to a fixed reference frame
- b) another difference from the expression in Eq. (2.2) is that  $s$  is not measured along a straight line

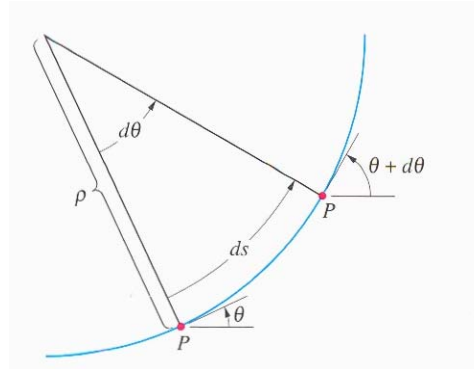
### 5.2 Acceleration of a particle in path coordinates

The acceleration  $\mathbf{a}$  of  $P$  can now be found by differentiating the expression in Eq. (5.3) with respect to time:

$$\mathbf{a} = \frac{d\mathbf{v}}{dt} = \frac{dv}{dt} \mathbf{e}_t + v \frac{d\mathbf{e}_t}{dt} \quad (5.4)$$

Because the unit vector  $\mathbf{e}_t$  changes direction  $\frac{d\mathbf{e}_t}{dt} \neq 0$ . Now we need to use the results of [Module 4](#) to determine the specific value of this derivative.

**Figure 2.** The instantaneous radius of curvature of the path of a particle



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According to Eq. (4.12) the time rate of change of  $\mathbf{e}_t$  is given by:

$$\frac{d\mathbf{e}_t}{dt} = \boldsymbol{\omega} \times \mathbf{e}_t \quad (5.5)$$

where  $\boldsymbol{\omega}$  must be the rate of rotation of  $\mathbf{e}_t$  relative to a fixed reference frame. In this case the rate of rotation of  $\mathbf{e}_t$  is due to the curvature of the path on which  $P$  travels. Since all motion and the rotation of  $\mathbf{e}_t$  takes place in the plane, the vector normal to the plane defines the direction of the angular velocity vector. Let this vector, which should be into or out of the plane of motion, be denoted by  $\mathbf{k}$ . Then  $\boldsymbol{\omega}$  is given by:

$$\boldsymbol{\omega} = \frac{d\theta}{dt} \mathbf{k}$$

where the quantity  $\frac{d\theta}{dt}$  can be determined from Fig. 2. As is seen from the figure, as  $P$  travels along the curve by a distance of  $ds$  the infinitesimal arc traveled subtends an angle  $d\theta$  which is equal to the infinitesimal change in the direction of the tangent unit vector  $\mathbf{e}_t$ . From basic geometry it is clear that

$$ds = \rho d\theta \quad (5.6)$$

where  $\rho$  is the *radius of curvature* of the path at that instant. The radius of curvature of a curve at  $P$  is the distance between  $P$  and the *center of curvature* of the curve at that point. For a general curve  $\rho$  changes from point to point on the curve. Dividing both sides by  $dt$  and rearranging we obtain:

$$\frac{d\theta}{dt} = \frac{1}{\rho} \frac{ds}{dt} = \frac{v}{\rho} \quad (5.7)$$

Thus the vector  $\boldsymbol{\omega}$  can be written as:

$$\boldsymbol{\omega} = \frac{v}{\rho} \mathbf{k} \quad (5.8)$$

Consequently we can write for the rate of change of  $\mathbf{e}_t$ :

$$\frac{d\mathbf{e}_t}{dt} = \frac{v}{\rho} \mathbf{k} \times \mathbf{e}_t \quad (5.9)$$

The vector product  $\mathbf{k} \times \mathbf{e}_t$  results in a new unit vector (since both components of the product are unit vectors) that is perpendicular to both  $\mathbf{k}$  and  $\mathbf{e}_t$ . This new vector, usually called the *normal unit vector*, is denoted by  $\mathbf{e}_n$ , and in this case is defined by:

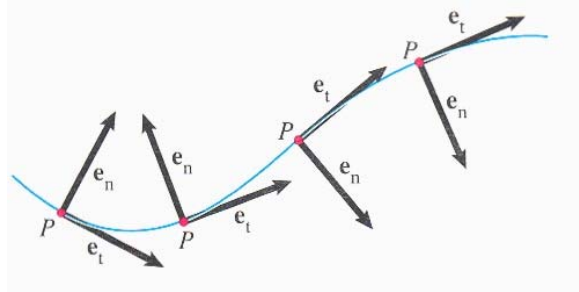
$$\mathbf{e}_n = \mathbf{k} \times \mathbf{e}_t \quad (5.10)$$

Note:

- a) the new vector is in the plane of motion (i.e. perpendicular to  $\mathbf{k}$ ) and along the radius of curvature (perpendicular to  $\mathbf{e}_t$ )

- b) by the right-hand-rule and the rules of the vector cross product operation  $\mathbf{e}_n$  points toward the center of curvature of the path at that position of  $P$  with  $\mathbf{e}_t$  pointing in the direction of the velocity of  $P$  (see Fig. 3)

**Figure 3.** The normal unit vector always points toward the center of curvature of the path of the particle



[Click to see an animated version of this figure](#)

Now the expression for the acceleration of  $P$  can be rewritten as:

$$\mathbf{a} = \frac{dv}{dt} \mathbf{e}_t + v \left( \frac{v}{\rho} \right) \mathbf{e}_n = \frac{dv}{dt} \mathbf{e}_t + \frac{v^2}{\rho} \mathbf{e}_n \quad (5.11)$$

This very important result is also often written as:

$$\mathbf{a} = a_t \mathbf{e}_t + a_n \mathbf{e}_n \quad (5.12)$$

with

$$a_t = \frac{dv}{dt} = \frac{d^2s}{dt^2} \quad a_n = \frac{v^2}{\rho} = \frac{1}{\rho} \left( \frac{ds}{dt} \right)^2 \quad (5.13)$$

In general  $a_t$  is known as the *tangential acceleration* and  $a_n$  as the *normal acceleration* of the particle.

If the path of  $P$  in the  $xy$  plane is described by a function  $y = y(x)$  then the instantaneous radius of curvature  $\rho$  at any point  $(x, y)$  on the curve is given by:

$$\rho = \frac{\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{\frac{3}{2}}}{\left(\frac{d^2y}{dx^2}\right)} \quad (5.14)$$

Note:

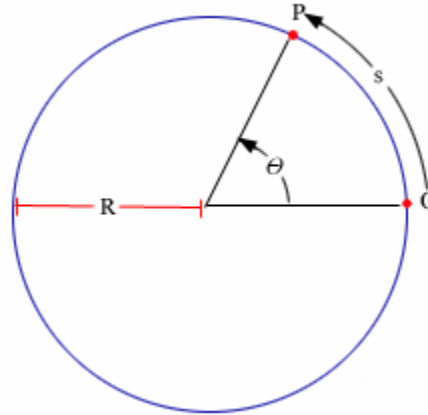
- a) the origin of the path coordinate system travels with  $P$  is always at  $P$
- b) the position of the particle is not easy to describe in path coordinates; it can only be specified in terms of  $s$ ; consequently these coordinates should be used when only velocity and acceleration are of interest
- c) a particle traveling along a curvilinear path has nonzero acceleration even if its speed along the path is constant; curvature of path always results in nonzero acceleration
- d) the radius of curvature of a straight path is infinite at all points (because  $\frac{d^2y}{dx^2} = 0$ ); consequently the tangential and normal acceleration components for a straight line path reduce to

$$a_t = \frac{d^2s}{dt^2} \quad a_n = 0$$

which are identical to the expressions obtained for rectilinear motion; in this case constant speed along the path would result in zero acceleration

### 5.3 Specialization to Circular Motion

**Figure 4.** Circular motion results in constant radius of curvature



When a particle travels along a perfectly circular path the radius of curvature of the path no longer varies from position to position. Specifically the radius of curvature of the path is a constant  $R$ . In this case Eq. (5.6) can be readily integrated as (see Fig. 4):

$$s = R\theta \quad (5.15)$$

resulting in:

$$\frac{ds}{dt} = R \frac{d\theta}{dt} = R\omega \quad \frac{d^2s}{dt^2} = R \frac{d^2\theta}{dt^2} = R\alpha \quad (5.16)$$

Thus Eqs. (5.13) reduce to the well known expressions:

$$a_t = \frac{d^2s}{dt^2} = R\alpha \quad a_n = \frac{1}{R} \left( \frac{ds}{dt} \right)^2 = R\omega^2 \quad (5.17)$$

Click here for Examples [1](#) and [2](#) on this topic