

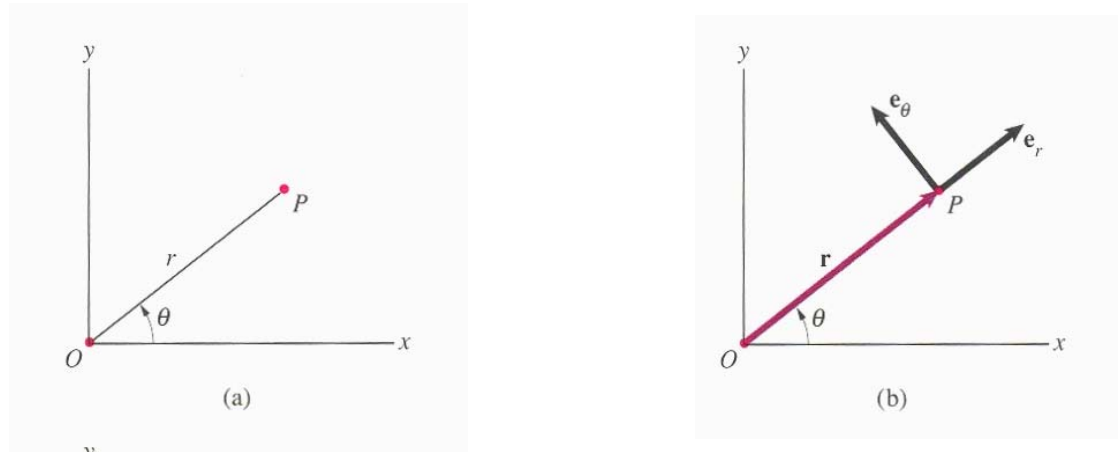
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سایت آموزش مهندسی مکانیک ایران

6. General Motion Described in Polar Coordinates

Certain classes of problems lend themselves to analysis in [polar or cylindrical coordinates](#). One important class of such problems is the motion of planets or satellites around the sun or the earth. In this section expressions are derived for the position, velocity, and acceleration of a particle in terms of polar coordinates.

Figure 1. a) Polar coordinates of a particle; b) the unit vectors \mathbf{e}_r and \mathbf{e}_θ and the position vector \mathbf{r}



[Click to see an animated version of this figure](#)

In polar coordinates the two scalar quantities used to define the position of a particle P in the xy plane are r and θ . (see Fig. 1) The first coordinate describes the radial distance of the particle from the origin O of the xy reference frame and the second defines the angle that the radial line from the origin to the particle makes with the positive x axis. In addition a triad of unit vectors (the coordinate system) is defined to track the particle as it moves in the plane. The unit vector \mathbf{e}_r always points from O to P . The second unit vector \mathbf{e}_θ also lies in the xy plane, is perpendicular to \mathbf{e}_r and points always in the direction of increasing θ . The third unit vector \mathbf{e}_z (or \mathbf{k}) points out of the paper such that:

$$\mathbf{e}_z = \mathbf{k} = \mathbf{e}_r \times \mathbf{e}_\theta$$

Because of these definitions both unit vectors \mathbf{e}_r and \mathbf{e}_θ rotate with angular velocity

$$\boldsymbol{\omega} = \frac{d\theta}{dt} \mathbf{e}_z = \frac{d\theta}{dt} \mathbf{k} = \dot{\theta} \mathbf{e}_z \quad (6.1)$$

while the unit vector \mathbf{e}_z remains fixed in space.

In terms of these quantities the position of P in the xy plane can be defined by:

$$\mathbf{r} = r \mathbf{e}_r \quad (6.2)$$

6.1. Velocity of a particle in polar coordinates

Then the velocity of P can be derived as:

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = \frac{dr}{dt} \mathbf{e}_r + r \frac{d\mathbf{e}_r}{dt} \quad (6.3)$$

Using the results of [Module 4](#) for the differentiation of rotating unit vectors we obtain:

$$\frac{d\mathbf{e}_r}{dt} = \boldsymbol{\omega} \times \mathbf{e}_r = \dot{\theta} \mathbf{k} \times \mathbf{e}_r = \dot{\theta} \mathbf{e}_\theta = \frac{d\theta}{dt} \mathbf{e}_\theta \quad (6.4)$$

where the last result is obtained by the rules of the vector cross product operation. Substituting this result into Eq. (6.3) we obtain for the velocity of P :

$$\begin{aligned} \mathbf{v} &= \frac{dr}{dt} \mathbf{e}_r + r \frac{d\theta}{dt} \mathbf{e}_\theta = \dot{r} \mathbf{e}_r + r \dot{\theta} \mathbf{e}_\theta \\ &= v_r \mathbf{e}_r + v_\theta \mathbf{e}_\theta \end{aligned} \quad (6.5)$$

In Eq. (6.5) the term $v_r = \dot{r}$ is known as the *radial component* and the term $v_\theta = r\dot{\theta}$ is known as the *transverse component* of velocity.

6.2 Acceleration of a particle in polar coordinates

The expression given by Eq. (6.5) can be differentiated with respect to time again to obtain the acceleration of P :

$$\mathbf{a} = \frac{d\mathbf{v}}{dt} = \frac{d^2r}{dt^2}\mathbf{e}_r + \frac{dr}{dt}\frac{d\mathbf{e}_r}{dt} + \frac{dr}{dt}\frac{d\theta}{dt}\mathbf{e}_\theta + r\frac{d^2\theta}{dt^2}\mathbf{e}_\theta + r\frac{d\theta}{dt}\frac{d\mathbf{e}_\theta}{dt} \quad (6.6)$$

Once again the derivative of \mathbf{e}_θ can be obtained using the results of Module 4:

$$\frac{d\mathbf{e}_\theta}{dt} = \boldsymbol{\omega} \times \mathbf{e}_\theta = \dot{\theta}\mathbf{k} \times \mathbf{e}_\theta = -\frac{d\theta}{dt}\mathbf{e}_r \quad (6.7)$$

where the last result is obtained by the rules of the vector cross product operation. Substituting the results of Eq. (6.4) and (6.7) into Eq. (6.6) and rearranging we obtain for the acceleration of P :

$$\begin{aligned} \mathbf{a} &= \frac{d\mathbf{v}}{dt} = \left[\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2 \right] \mathbf{e}_r + \left[r\frac{d^2\theta}{dt^2} + 2\frac{dr}{dt}\frac{d\theta}{dt} \right] \mathbf{e}_\theta \\ &= (\ddot{r} - r\dot{\theta}^2)\mathbf{e}_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\mathbf{e}_\theta \end{aligned} \quad (6.8)$$

This result is often written as:

$$\mathbf{a} = a_r\mathbf{e}_r + a_\theta\mathbf{e}_\theta \quad (6.9)$$

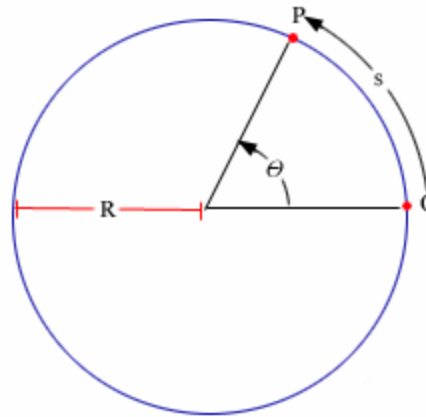
with

$$\begin{aligned} a_r &= \ddot{r} - r\dot{\theta}^2 \\ a_\theta &= r\ddot{\theta} + 2\dot{r}\dot{\theta} \end{aligned} \quad (6.10)$$

where a_r is known as the *radial component* and a_θ as the *transverse component* of acceleration. The term $-r\dot{\theta}^2$ is called the *centripetal acceleration* and is always directed toward origin of the coordinates, and the term $2r\dot{\theta}$ is called the *Coriolis acceleration*.

6.3 Specialization to Circular Motion

Figure 2. Circular motion results in a constant radial distance from the origin



When a particle travels along a perfectly circular path centered around the origin of the xy reference frame its radial distance from the center remains at the constant value of R . Thus we obtain:

$$r = R \quad \dot{r} = 0 \quad \ddot{r} = 0 \quad (6.11)$$

which modify Eqs. (6.5) and (6.10) to:

$$\mathbf{v} = R\dot{\theta}\mathbf{e}_\theta \quad v = R\dot{\theta} \quad (6.12)$$

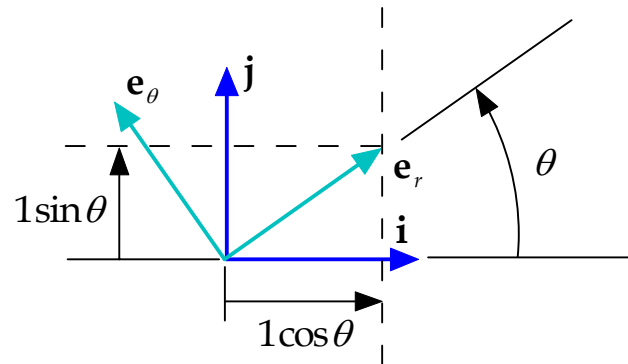
$$a_r = -R\dot{\theta}^2 = -\frac{v^2}{R} \quad a_\theta = R\ddot{\theta} \quad (6.13)$$

Note:

- in circular motion expressions for the components velocity and acceleration in polar and path coordinates are exactly equivalent; compare Eqs. (6.12), (6.13) to Eqs. (5.16) and (5.17)
- nevertheless the origins of the two coordinate systems are different; the origin of the path coordinate system is always at P whereas the origin of the polar coordinate system is at O ; hence the difference in sign between a_n and a_r .

6.4 Relationship to Cartesian coordinates

Figure 3.
Relationship
between polar and
Cartesian unit
vectors



Polar and Cartesian coordinates are readily convertible into each other. Consider a particle P whose position in the xy plane is equivalently expressed by the Cartesian coordinates x and y and the polar coordinates r and θ . Then using simple trigonometry we can write:

$$\begin{aligned} x &= r \cos \theta \\ y &= r \sin \theta \end{aligned} \quad (6.14)$$

and

$$\begin{aligned}r &= \sqrt{x^2 + y^2} \\ \theta &= \arctan\left(\frac{y}{x}\right)\end{aligned}\tag{6.15}$$

These expressions may be very useful in circumstances where conversions between two coordinate systems are required.

The unit vectors associated with polar coordinates can also be transformed to unit vectors associated with Cartesian coordinates. From Fig. 3 it is clear that

$$\begin{aligned}\mathbf{e}_r &= \cos \theta \mathbf{i} + \sin \theta \mathbf{j} \\ \mathbf{e}_\theta &= -\sin \theta \mathbf{i} + \cos \theta \mathbf{j}\end{aligned}$$

Conversely

$$\begin{aligned}\mathbf{i} &= \cos \theta \mathbf{e}_r - \sin \theta \mathbf{e}_\theta \\ \mathbf{j} &= \sin \theta \mathbf{e}_r + \cos \theta \mathbf{e}_\theta\end{aligned}$$

Click here for Examples [1](#), [2](#), and [3](#) on this topic