8. Energy and Momentum Concepts

The direct application of Newton's 2nd Law may sometimes be unwieldy or burdensome for the calculation of desired quantities. This is particularly true in circumstances where the path of the particle subjected to a force is complicated but immaterial to the problem or when the applied force is complex but its exact nature is not required for calculating a desired value. In such circumstances advantage can be taken of other scalar or vector dynamic quantities such energy and momentum.

8.1 Definitions of Dynamic Quantities

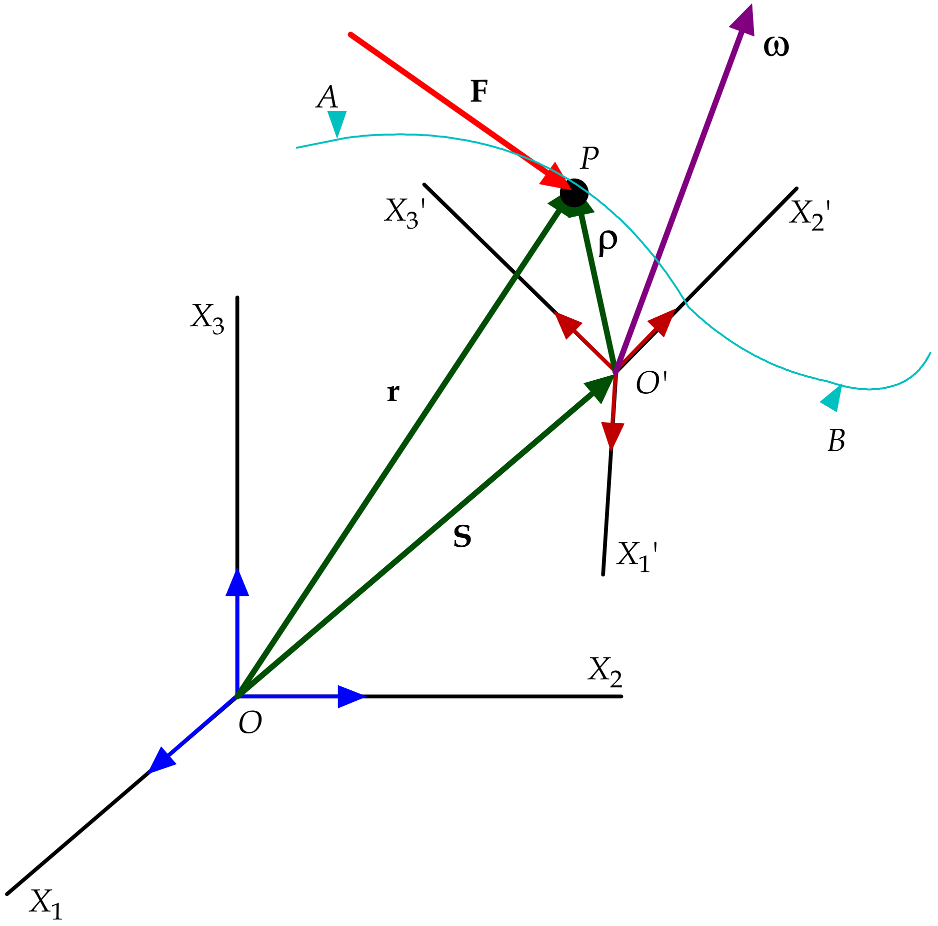

Figure 1. A particle moving along a path subjected to a resultant force; position of the particle is defined relative to two reference frames, one fixed and one moving

Consider a particle P of mass m in Cartesian space subjected to a force **F**. Let the position of the particle relative to fixed (inertial) and moving coordinates be defined by the vectors **r** and $\boldsymbol{\rho}$, respectively. The particle is assumed to be at A at time t_A and at B at time t_B.

The following dynamic quantities can now be defined:

The total impulse of the force **F** over the time interval $\Delta t = t_B - t_A$ is defined as:

$$\mathcal{F} \triangleq \int_{t_A}^{t_B} \mathbf{F}\,dt$$

The linear momentum of the particle is defined as:

$$\mathbf{G} \triangleq m\mathbf{v}$$

where **v** is the velocity of the particle at any time.

The angular momentum of the particle about O is defined as:

$$\mathbf{H}_O = \mathbf{H} \triangleq \mathbf{r} \times m\mathbf{v}$$

(also known as moment of momentum).

The angular momentum of the particle about O' is similarly defined as:

$$\mathbf{H}_{O'} = \boldsymbol{\rho} \times m\dot{\boldsymbol{\rho}}$$

Note that angular momentum is always taken about a specific point.

The moment of the force **F** about O is defined as:

$$\mathbf{M}_O = \mathbf{M} \triangleq \mathbf{r} \times \mathbf{F}$$

The moment of the force **F** about O' is similarly defined as:

$$\mathbf{M}_{O'} \triangleq \boldsymbol{\rho} \times \mathbf{F}$$

The work done by the force **F** in moving the particle from A to B is defined as:

$$W \triangleq \int_A^B \mathbf{F} \cdot d\mathbf{r}$$

where the integral must be evaluated along the path of P.

The kinetic energy of the particle is defined as:

$$T \triangleq \frac{1}{2} m\mathbf{v} \cdot \mathbf{v} = \frac{1}{2} m v^2$$

8.2 Relationships Resulting from Newton's 2nd Law

Linear Impulse-Momentum Relation

For the particle considered above we can write Newton's 2nd Law as:

$$\mathbf{F} = m\ddot{\mathbf{r}} = m\frac{d\mathbf{v}}{dt} \tag{8.1}$$

or

$$\mathbf{F}\,dt = m\,d\mathbf{v}$$

Integrating the two sides of this equation we obtain:

$$\int_{t_A}^{t_B} \mathbf{F}\, dt = \int_A^B m\, d\mathbf{v} = m\left(\mathbf{v}_B - \mathbf{v}_A\right) \tag{8.2}$$

Eq. (8.2) states that the <u>total impulse of a force</u> acting on a particle over a time interval <u>equals the change in the linear momentum</u> of the particle over the same interval. Note that this is a vector equation, and if $F_i = 0$ for any $i = 1,2,3$ $v_{Bi} = v_{Ai}$. In such instances the linear momentum of the particle is said to have been conserved in the given direction.

<u>Work-Energy Relation</u>

Taking the dot product of both sides of Eq. (8.1) with **v** we obtain:

$$\mathbf{F}\cdot\mathbf{v} = \mathbf{F}\cdot\frac{d\mathbf{r}}{dt} = m\frac{d\mathbf{v}}{dt}\cdot\mathbf{v}$$

which can also be written as:

$$\mathbf{F}\cdot d\mathbf{r} = m\mathbf{v}\cdot d\mathbf{v} = \frac{1}{2}m\, d\left(\mathbf{v}\cdot\mathbf{v}\right)$$

Integrating both sides of this equation

$$\int_A^B \mathbf{F}\cdot d\mathbf{r} = \frac{1}{2}m\int_A^B d\left(\mathbf{v}\cdot\mathbf{v}\right) = \frac{1}{2}m\left(v_B^2 - v_A^2\right) = T_B - T_A \tag{8.3}$$

Eq. (8.3) states that the <u>work done by a force</u> acting on a particle over the path of the particle <u>equals the change in the kinetic energy</u> of the particle over the same path. Note that this is a scalar equation.

In instances where $\mathbf{F}\cdot d\mathbf{r}=0$ for a given force and path $T_B=T_A$. In such instances the kinetic energy of the particle is said to have been conserved.

<u>Moment-Angular Momentum Relation</u>

Taking the cross product of both sides of Eq. (8.1) with **r** we obtain:

$$\mathbf{r}\times\mathbf{F}=\mathbf{r}\times m\frac{d\mathbf{v}}{dt}$$

However,

$$\begin{aligned}\frac{d}{dt}(\mathbf{r}\times m\mathbf{v})&=\frac{d\mathbf{r}}{dt}\times m\mathbf{v}+\mathbf{r}\times m\frac{d\mathbf{v}}{dt}\\&=\mathbf{v}\times m\mathbf{v}+\mathbf{r}\times m\frac{d\mathbf{v}}{dt}=\mathbf{r}\times m\frac{d\mathbf{v}}{dt}\end{aligned}$$

Thus

$$\mathbf{r}\times\mathbf{F}=\mathbf{r}\times m\frac{d\mathbf{v}}{dt}=\frac{d}{dt}(\mathbf{r}\times m\mathbf{v})$$

Using the definition of moment and angular momentum this relation can be written as;

$$\mathbf{M}=\frac{d\mathbf{H}}{dt}=\dot{\mathbf{H}} \tag{8.4}$$

where the subscript O has been dropped for brevity and is deduced from the moment arm of the force. Eq. (8.4) states that the <u>moment</u> of a force acting on a particle about a point <u>equals the rate of change of the angular momentum</u> of the particle about the same point. Note that this is a vector equation and if $M_i=0$ for any $i=1,2,3$ $H_i=\text{constant}$.

Angular Impulse-Angular Momentum Relation

Eq. (8.4) can also be written as:

$$\mathbf{M}\,dt = d\mathbf{H}$$

Integrating both sides of this equation we obtain:

$$\int_{t_A}^{t_B} \mathbf{M}\,dt = \mathcal{M} = \int_{A}^{B} d\mathbf{H} = \mathbf{H}_B - \mathbf{H}_A \qquad (8.5)$$

The quantity $\mathcal{M}$ is known as the total angular impulse of the force **F** about the origin of the reference frame over the time interval $\Delta t = t_B - t_A$. Eq. (8.5) states that the total angular impulse of a force acting on a particle about any point over a time interval equals the change in the angular momentum of the particle about the same point over the same time interval. Note that this is a vector equation, and if $M_i = 0$ for any $i = 1,2,3$ $H_{Bi} = H_{Ai}$. In such instances the angular momentum of the particle is said to have been conserved in the given direction.

8.3 Conservative Forces

A force **F** is said to be conservative if:

a) $\mathbf{F} = \mathbf{F}(\mathbf{r})$ (i.e. the force is a function of position only and not of time or velocity); and

b) $\int_{A}^{B} \mathbf{F}\cdot d\mathbf{r} = -\int_{B}^{A} \mathbf{F}\cdot d\mathbf{r}$ (i.e. the work done by the force does not depend on the path taken and depends only on the end points)

The second relation above implies that $\oint \mathbf{F}\cdot d\mathbf{r} = 0$, i.e. the work done by a conservative force around a closed loop path is zero. Note also that the first requirement is a necessary condition for the second requirement because if **F** did not depend on position only it would be impossible to ensure that the work done by the force depends only on the end points. If **F** depended on, say, time, the work done in going from one point to another would also have a time component which means that the second requirement could not hold.

If **F** is conservative a scalar quantity called potential or potential energy is defined as:

$$\mathbf{F}\cdot d\mathbf{r} \triangleq -dV$$

where the negative sign is placed by convention. Now the work done by **F** in moving the particle from point A to B along any path can be written as a change in the potential energy:

$$\int_A^B \mathbf{F}\cdot d\mathbf{r} = -\int_A^B dV = V_A - V_B \qquad (8.6)$$

Combining Eq. (8.6) with Eq. (8.3) we obtain for a conservative force acting on a particle:

$$V_A - V_B = T_B - T_A$$

or

$$V_A + T_A = V_B + T_B \qquad (8.7)$$

The sum of potential and kinetic energy is known as total mechanical energy. Eq. (8.7) expresses the principle of conservation of mechanical energy in the presence of conservative forces.
Some common examples of the potential energy function are the energy stored in a spring, the potential energy of a particle near or away from the earth's surface, and the potential energy of an electrically charged particle in an electromagnetic field. The energy stored in a linear spring of stiffness k deflected by a distance δ from its unstretched length is given by:

$$V_s = \frac{1}{2}k\delta^2$$

The potential energy of a particle in the earth's gravitational field is given by Newton's Gravitational Law. Click here for a derivation of gravitational potential energy near earth's surface.

Click here for Examples 1, 2, 3, and 4 on this topic.