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سایت آموزش مهندسی مکانیک ایران

9. Impact and Collision

One major class of interaction between two particles is collision. Although the treatment given below is for the collision of two small smooth spheres moving on a plane the concepts developed can be used in the analysis of more complex problems.

9.1 Idealization of Impact

Consider two smooth spheres (particles) moving on a plane with velocities $\mathbf{v}_1$ and $\mathbf{v}_2$ just before they collide with each other some time t_0. We assume that the mass centers of the spheres coincide with the centers of curvature of the surfaces coming in contact. We also assume that contact between the two particles exists for an infinitesimal period of time $(t_0^+ - t_0)$ so that the particles have no time to change their positions but that their velocities change instantaneously as shown in Fig. 1.

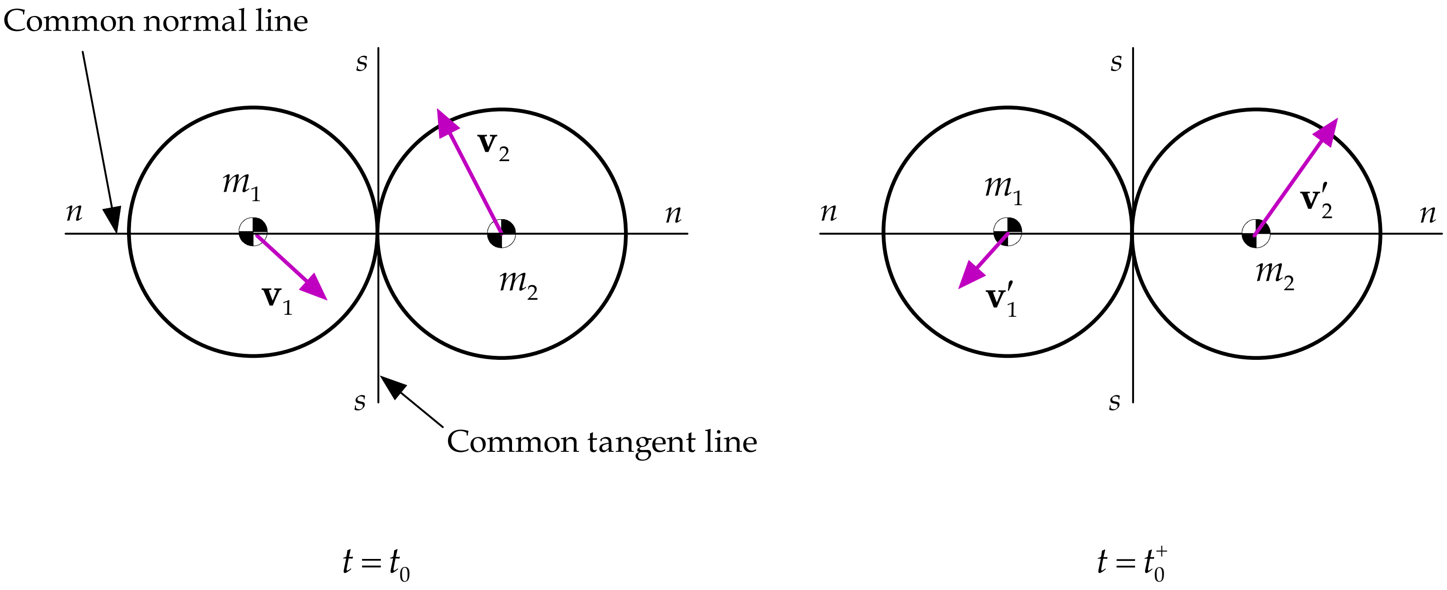

Figure 1. Two colliding particles at $t = t_0$

We further assume that a force of interaction develops normal to the common tangent plane at the point of contact or along the line joining the centers of curvature of the two particles at the point of contact. We assume that the force of interaction occurs over an infinitesimal period of time. Under these assumptions the velocities of the particles change instantaneously in the normal direction (*n*-*n*) and remain unchanged along the common tangent (*s*-*s*). This is depicted in Fig. 2.

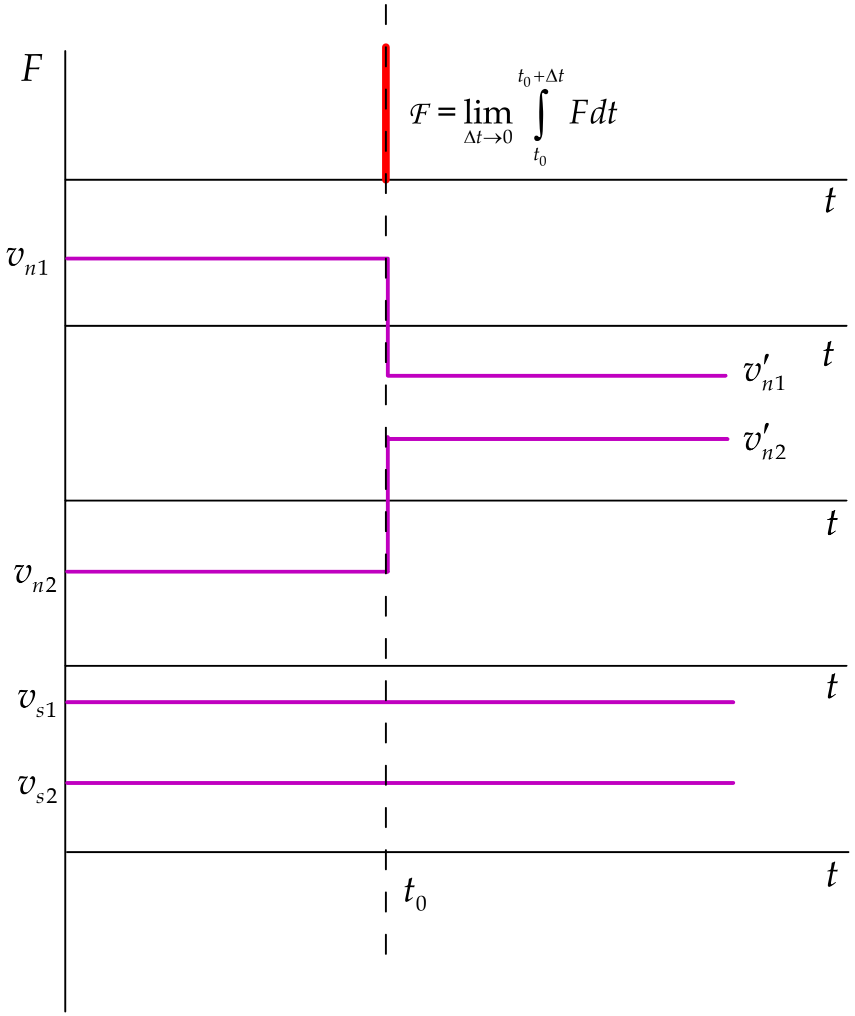

Figure 2. Instantaneous changes in velocities in the direction of impact at $t = t_0$

Under the assumptions made it is clear that the force of interaction between the two particles exists only for an infinitesimal interval of time. The finite impulse $\mathcal{F}$ generated by the force of interaction F during the infinitesimal interval of time is defined by:

$$\mathcal{F} = \lim_{\Delta t \to 0} \int_{t_0}^{t_0+\Delta t} F\,dt$$

Note that a finite impulse developing over infinitesimal time implies that a force of infinite magnitude exists for infinitesimal time. This type of force is also known as an *impulsive force*. The free-body impulse balance diagrams can be drawn for the two particles as shown in Fig. 3.

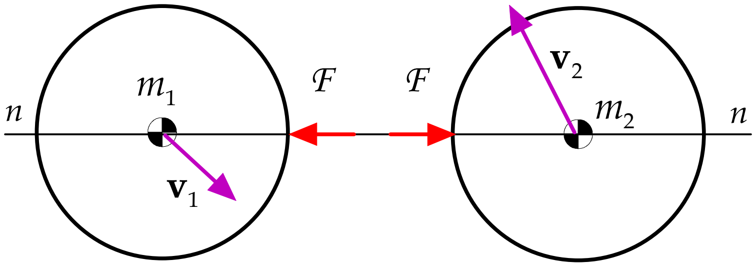

Figure 3. Free-body impulse balance diagram for two colliding particles

The linear impulse-momentum equations can now be written for the two particles:

$$\begin{aligned} -\mathcal{F} &= m_1\left(v'_{n1} - v_{n1}\right) \\ \mathcal{F} &= m_2\left(v'_{n2} - v_{n2}\right) \end{aligned} \tag{1}$$

Note that if these two equations are summed we obtain:

$$m_1\left(v'_{n1} - v_{n1}\right) + m_2\left(v'_{n2} - v_{n2}\right) = 0 \tag{2}$$

or

$$m_1 v'_{n1} + m_2 v'_{n2} = m_1 v_{n1} + m_2 v_{n2} \tag{3}$$

which states that the total linear momentum of the two particles in the normal direction is conserved. This is expected since there are no external impulsive forces acting on the system; the force of interaction between the two particles is an internal force.

Since there are no impulsive forces acting on either particle in the common tangential direction we can write:

$$\begin{aligned} v'_{s1} &= v_{s1} \\ v'_{s2} &= v_{s2} \end{aligned} \tag{4}$$

The four relations in Eqs. (1) and (4) are not sufficient to solve for the velocities of the particles after impact and the resulting impulse. An additional parameter is needed to characterize the impact between the two particles in terms of the velocities of the particles before and after impact. This parameter e, known as the *coefficient of restitution*, is defined by the relation:

$$e = \frac{v'_{n2} - v'_{n1}}{v_{n1} - v_{n2}} \tag{5}$$

The coefficient of restitution for a pair of colliding spheres is a dimensionless number between 0 and 1 and depends on the material properties, shape, size, impact velocities of the spheres and can be experimentally determined. Solving Eqs. (1) and (5) for the post-impact velocities and the resulting impulse, we obtain:

$$v'_{n1} = \frac{1}{m_1 + m_2}\left[\left(m_1 - e m_2\right) v_{n1} + m_2 \left(1 + e\right) v_{n2}\right] \tag{6}$$

$$v'_{n2} = \frac{1}{m_1 + m_2}\left[m_1 \left(1 + e\right) v_{n1} + \left(m_2 - e m_1\right) v_{n2}\right] \tag{7}$$

$$\mathcal{F} = \frac{m_1 m_2}{m_1 + m_2}(1+e)(v_{n1} - v_{n2}) \tag{8}$$

9.2 Change in Kinetic Energy

The total kinetic energy of the particle pair before impact can be represented as:

$$T = T_C + T_s + T_n \tag{9}$$

where $T_C = \frac{1}{2}(m_1 + m_2)v_C^2$ is the kinetic energy of the particles referred to their center of mass; T_s is the kinetic energy due to the tangential components of the velocities relative to the center of mass; and T_n is the kinetic energy due to the normal components of the velocities relative to the center of mass. The total kinetic energy of the particles after impact is given by:

$$T' = T'_C + T'_s + T'_n \tag{10}$$

Since there are no external impulsive forces acting on the pair, the velocity of the center of mass does not change after impact. Also according to Eq. (4) the velocities of the particles in the tangential direction do not change. As a result the first two components kinetic energy in Eq. (9) remain unchanged after the impact:

$$\begin{aligned} T'_C &= T_C \\ T'_s &= T_s \end{aligned} \tag{11}$$

The normal components of the velocities of the two particles velocities relative to the center of mass before and after impact can be written as:

$$\begin{aligned} (v_{ni})_r &= v_{ni} - v_{Cn} \\ (v'_{ni})_r &= v'_{ni} - v_{Cn} \end{aligned} \qquad i = 1,2 \tag{12}$$

where v_{Cn} is the normal component of the velocity of the center of mass of the pair. The third component of kinetic energy in Eqs. (9) and (10) is given by:

$$\begin{aligned} T_n &= \frac{1}{2}\left[m_1\left(v_{n1}\right)_r^2 + m_2\left(v_{n2}\right)_r^2\right] \\ T_n' &= \frac{1}{2}\left[m_1\left(v_{n1}'\right)_r^2 + m_2\left(v_{n2}'\right)_r^2\right] \end{aligned} \tag{13}$$

Substituting the relations given by Eq. (12) in Eq. (5) and the resulting relations together with Eqs. (6) and (7) in the relations given by Eq. (13) and after extensive algebraic manipulation we obtain:

$$T_n' = e^2 T_n \tag{14}$$

Thus for a general value of e the total kinetic energy of the particles is reduced after impact and the reduction is expressed by Eq. (14).

9.3 Special cases

a) Perfectly plastic impact ($e = 0$)

From Eqs. (6) and (7) it is clear that when $e = 0$

$$v_{n1}' = v_{n2}' = \frac{1}{m_1 + m_2}\left(m_1 v_{n1} + m_2 v_{n2}\right) \tag{15}$$

meaning that the two particles travel or stick together after impact. This situation is known as *perfectly plastic or inelastic impact*. In this case, from Eq. (14):

$$T_n' = 0$$

meaning that

$$T' = T'_C + T'_s = T_C + T_s \tag{16}$$

Thus there is no conservation of energy and all of T_n is lost in a plastic impact.

b) Perfectly elastic impact ($e = 1$)

When $e = 1$ Eqs. (6) and (7) yield:

$$v'_{n1} = \frac{1}{m_1 + m_2}\left[(m_1 - m_2)v_{n1} + 2m_2 v_{n2}\right] \tag{17}$$

$$v'_{n2} = \frac{1}{m_1 + m_2}\left[2m_1 v_{n1} + (m_2 - m_1)v_{n2}\right] \tag{18}$$

Also from Eq. (14)

$$T'_n = T_n$$

which means

$$T' = T'_C + T'_s + T'_n = T_C + T_s + T_n = T \tag{19}$$

Thus energy is fully conserved.

c) Impact with large, stationary mass ($m_2 \gg m_1$, $v_{n2} = 0$)

In cases of impact with large stationary objects such as a wall or the ground we assume that $m_2 \gg m_1$ and $v_{n2} = 0$ in Eqs. (6) and (7). For $v_{n2} = 0$ these two equations can be written as:

$$v'_{n1} = \frac{1}{m_1 + m_2}\left[\left(m_1 - em_2\right)v_{n1}\right] = \left(\frac{m_1 / m_2}{m_1 / m_2 + 1} - \frac{e}{m_1 / m_2 + 1}\right)v_{n1}$$

$$v'_{n2} = \frac{1}{m_1 + m_2}\left[m_1\left(1+e\right)v_{n1}\right] = \left(\frac{m_1 / m_2}{m_1 / m_2 + 1}\right)\left(1+e\right)v_{n1}$$

If $m_2 \gg m_1$, $m_1 / m_2 \approx 0$, which results in:

$$v'_{n1} \approx -ev_{n1} \tag{20}$$

$$v'_{n2} \approx 0 \tag{21}$$

In this case the change in the total kinetic energy is expressed by the change in the kinetic energy of the moving smaller particle:

$$T' = e^2 T \tag{22}$$

meaning that there is no conservation of energy except if $e = 1$.

The impulse generated by this type of impact is given by:

$$\mathcal{F} = m_1\left(v'_{n1} - v_{n1}\right) = -m_1\left(1+e\right)v_{n1} \tag{23}$$

9.4 Effect of Friction

If Coulomb friction exists at the point of contact between the two spheres and is defined by a coefficient of friction μ then we assume that a friction impulse develops along the common tangent to the two contacting surfaces. In this case the free-body impulse-momentum diagram for the two spheres becomes as shown in Fig. 4.

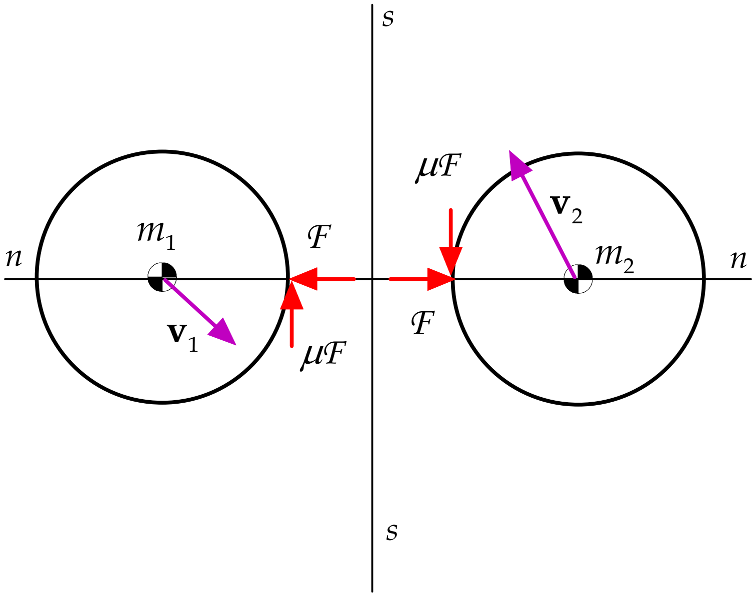

Figure 4. Free-body impulse balance diagram for two colliding particles in the presence of friction

The new impulsive forces result in changes in the velocities of the particles along the common tangent to the contacting surfaces and Eqs. (4) need to be modified to

$$\begin{aligned} \mu\mathcal{F} &= m_1\left(v'_{s1} - v_{s1}\right) \\ -\mu\mathcal{F} &= m_2\left(v'_{s2} - v_{s2}\right) \end{aligned} \tag{24}$$

where the impulse $\mathcal{F}$ is still given by Eq. (8). Eqs. (24) can now be solved for the tangential velocity components after impact:

$$v'_{s1} = v_{s1} + \frac{\mu m_2}{m_1 + m_2}(1+e)\left(v_{n1} - v_{n2}\right) \tag{25}$$

$$v'_{s2} = v_{s2} + \frac{\mu m_1}{m_1 + m_2}(1+e)\left(v_{n1} - v_{n2}\right) \tag{26}$$

Click here for an example on impact resulting in linear and angular momentum changes.