Dynamics of a System of Particles

Consider a system of N particles that are in general motion in Cartesian space.

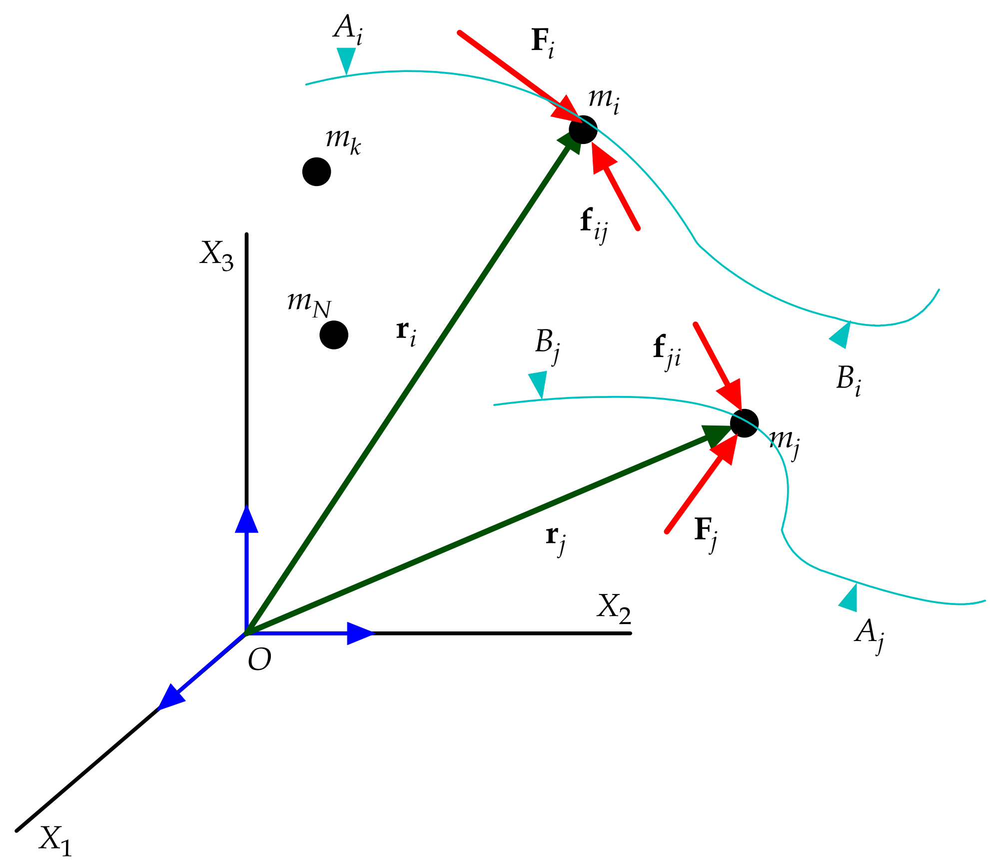

Figure 1. A system of N particles in Cartesian space

In Fig. 1:

$\mathbf{r}_i$ is the vector defining the position of particle i, which has mass m_i, in inertial space
$\mathbf{F}_i$ is the resultant external force acting on particle i

$\mathbf{f}_{ij}$ is the force exerted by particle j on particle i

A_i and B_i are locations of particle i at times t_A and t_B, respectively, on the path of particle i

We stipulate that the interparticle forces lie along straight lines that connect pairs of particles, are equal and opposite to each other, and that no particle can exert any force on itself. Thus

$$\mathbf{f}_{ij} = -\mathbf{f}_{ji} \quad \mathbf{f}_{ii} = \mathbf{0} \quad \boldsymbol{\rho}_{ij} \times \mathbf{f}_{ij} = \mathbf{0} \quad i, j = 1, \ldots, N \tag{1}$$

where $\boldsymbol{\rho}_{ij}$ is the vector from particle j to particle i.

Writing Newton's 2nd Law for particle i we obtain:

$$\mathbf{F}_i + \sum_{j=1}^{N} \mathbf{f}_{ij} = m_i \ddot{\mathbf{r}}_i \tag{2}$$

If we sum up this relation over i we obtain:

$$\sum_{i=1}^{N} \mathbf{F}_i + \sum_{i=1}^{N} \sum_{j=1}^{N} \mathbf{f}_{ij} = \sum_{i=1}^{N} m_i \ddot{\mathbf{r}}_i$$

However $\sum_{i=1}^{N} \sum_{j=1}^{N} \mathbf{f}_{ij} = \mathbf{0}$ because $\mathbf{f}_{ij} = -\mathbf{f}_{ji}$ and $\mathbf{f}_{ii} = \mathbf{f}_{jj} = \mathbf{0}$. Thus:

$$\sum_{i=1}^{N} \mathbf{F}_i = \sum_{i=1}^{N} m_i \ddot{\mathbf{r}}_i \tag{3}$$

Definitions:

The net resultant external force on the system is defined as:

$$\mathbf{F} \triangleq \sum_{i=1}^{N} \mathbf{F}_i$$

The total mass of the particles is defined as:

$$m \triangleq \sum_{i=1}^{N} m_i$$

The position of the center of mass of the particles is defined as:

$$\mathbf{r}_C \triangleq \frac{1}{m} \sum_{i=1}^{N} m_i \mathbf{r}_i$$

With these definitions Eq. (3) can be written as:

$$\mathbf{F} = m\ddot{\mathbf{r}}_C \tag{4}$$

which states that the net external resultant force acting on a system of particles equals the total mass of the particles multiplied by the acceleration of the center of mass of the particles.

The definitions above introduce a new point C to the configuration of particles as shown in Fig. 2. The motion of this point over time provides significant information about the motion of the entire system and can be used to describe the overall motion of the system in concise form.

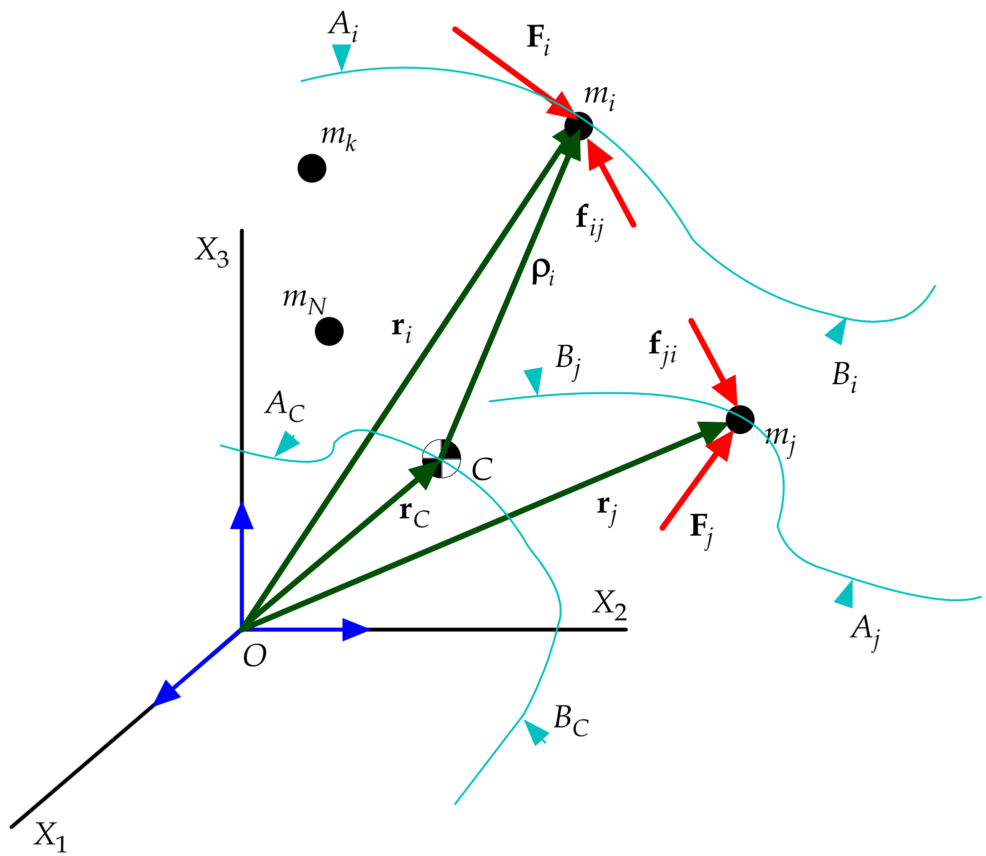

Figure 2. Relation of center of mass to other particles

Linear Impulse-Momentum Relation

Eq. (4) can be written as:

$$\mathbf{F} = m\frac{d\mathbf{v}_C}{dt}$$

or

$$\mathbf{F}\,dt = m\,d\mathbf{v}_C$$

Integrating the two sides of this equation we obtain:

$$\int_{t_A}^{t_B} \mathbf{F}\,dt = \boldsymbol{\mathcal{F}} = m\int_A^B d\mathbf{v}_C = m\left(\mathbf{v}_{CB} - \mathbf{v}_{CA}\right) \tag{5}$$

Eq. (5) states that the total impulse of the resultant of the external forces acting on a system of particles over a time interval equals the change in the linear momentum of the particles referred to the center of mass over the same interval. Note that this is a vector equation, and if $F_i = 0$ for any $i = 1,2,3$ $v_{CBi} = v_{CAi}$. In such instances the linear momentum of the particles is said to have been conserved in the given direction.

Note also that from the definition of the position of the center of mass

$$m\mathbf{v}_C = m\dot{\mathbf{r}}_C = \sum_{i=1}^{N} m_i \dot{\mathbf{r}}_i = \sum_{i=1}^{N} \mathbf{p}_i$$

which means that the linear momentum of the particles referred to the center of mass equals the total linear momentum of the particles.

Work-Energy Relations

Taking the dot product of the two sides of Eq. (4) with $\mathbf{v}_C$ we obtain:

$$\mathbf{F}\cdot\mathbf{v}_C = m\frac{d\mathbf{v}_C}{dt}\cdot\mathbf{v}_C$$

or

$$\mathbf{F} \cdot \frac{d\mathbf{r}_C}{dt} = m\mathbf{v}_C \cdot \frac{d\mathbf{v}_C}{dt}$$

or

$$\mathbf{F} \cdot d\mathbf{r}_C = m\mathbf{v}_C \cdot d\mathbf{v}_C = \frac{1}{2} md\left(\mathbf{v}_C \cdot \mathbf{v}_C\right) = \frac{1}{2} md\left(v_C^2\right) \tag{6}$$

Integrating both sides of this equation we obtain:

$$\int_{A_C}^{B_C} \mathbf{F} \cdot d\mathbf{r}_C = \frac{1}{2} m \int_A^B d\left(v_C^2\right) = \frac{1}{2} m\left(v_{CB}^2 - v_{CA}^2\right) = T_{CB} - T_{CA} \tag{7}$$

Eq. (7) states that the work done by the resultant of the external forces in moving the center of mass along its path between A_C and B_C (see Fig. 2) equals the change in the kinetic energy of the particles referred to the center of mass. Note that the integral on the left side of Eq. (7) is not the total work done by all the forces in the system in moving all the particles along their paths, and the right side of the equation is not the change in the total kinetic energy of the particles. Also note that this is a scalar equation. In instances where $\mathbf{F} \cdot d\mathbf{r}_C = 0$ for a given resultant force and path $T_{CB} = T_{CA}$.

To determine the relationship between the total work done by all the forces (external and internal) in moving the particles along their paths and the change in the total kinetic energy of the particles consider Eq. (2):

$$\mathbf{F}_i + \sum_{j=1}^{N} \mathbf{f}_{ij} = m_i \ddot{\mathbf{r}}_i = m_i \dot{\mathbf{v}}_i$$

Using the same arguments utilized to develop Eq. (6) we can write:

$$\left(\mathbf{F}_i + \sum_{j=1}^{N} \mathbf{f}_{ij}\right) \cdot d\mathbf{r}_i = m_i \mathbf{v}_i \cdot d\mathbf{v}_i \tag{8}$$

From Fig. 2, we can write for the position of particle *i*:

$$\mathbf{r}_i = \mathbf{r}_C + \boldsymbol{\rho}_i$$

The work done by all the forces in moving particle *i* along its path is given by:

$$W_i = \int_{A_i}^{B_i} \left(\mathbf{F}_i + \sum_{j=1}^{N} \mathbf{f}_{ij}\right) \cdot d\mathbf{r}_i = \int_{A_i}^{B_i} \left(\mathbf{F}_i + \sum_{j=1}^{N} \mathbf{f}_{ij}\right) \cdot \left(d\mathbf{r}_C + d\boldsymbol{\rho}_i\right)$$

The work done by all the forces in moving all the particles along their paths is obtained by summing these individual work terms over *i*:

$$\begin{aligned} W = \sum_{i=1}^{N} W_i &= \sum_{i=1}^{N} \int_{A_i}^{B_i} \left(\mathbf{F}_i + \sum_{j=1}^{N} \mathbf{f}_{ij}\right) \cdot \left(d\mathbf{r}_C + d\boldsymbol{\rho}_i\right) \\ &= \sum_{i=1}^{N} \int_{A_C}^{B_C} \left(\mathbf{F}_i + \sum_{j=1}^{N} \mathbf{f}_{ij}\right) \cdot d\mathbf{r}_C + \sum_{i=1}^{N} \int_{A_i}^{B_i} \left(\mathbf{F}_i + \sum_{j=1}^{N} \mathbf{f}_{ij}\right) \cdot d\boldsymbol{\rho}_i \end{aligned}$$

Note that the limits of the integral in the first term are changed because the integration is performed over the path of *C*, the center of mass. In this expression the summation and integral signs can be interchanged if the limits of the integrals are not functions of the summation index *i*. Thus

$$W = \int_{A_C}^{B_C} \left(\sum_{i=1}^{N} \mathbf{F}_i\right) \cdot d\mathbf{r}_C + \int_{A_C}^{B_C} \left(\sum_{i=1}^{N} \sum_{j=1}^{N} \mathbf{f}_{ij}\right) \cdot d\mathbf{r}_C + \sum_{i=1}^{N} \int_{A_i}^{B_i} \left(\mathbf{F}_i + \sum_{j=1}^{N} \mathbf{f}_{ij}\right) \cdot d\boldsymbol{\rho}_i$$

The second term in this expression is zero by the same arguments as those used for Eq. (3). This results in:

$$W = \int_{A_C}^{B_C} \mathbf{F} \cdot d\mathbf{r}_C + \sum_{i=1}^{N} \int_{A_i}^{B_i} \left(\mathbf{F}_i + \sum_{j=1}^{N} \mathbf{f}_{ij} \right) \cdot d\boldsymbol{\rho}_i \tag{9}$$

The first term in this expression is identical to the left side of Eq. (7) and has the same meaning. The second term represents the work done by the external and internal forces in moving the particles along paths relative to the center of mass.

Looking at the right side of Eq. (8) we can write:

$$\begin{aligned} W_i &= m_i \int_{A_i}^{B_i} \mathbf{v}_i \cdot d\mathbf{v}_i = \frac{1}{2} m_i \left(\mathbf{v}_i \cdot \mathbf{v}_i \right) \Big|_{A_i}^{B_i} \\ &= \frac{1}{2} m_i \left[\left(\dot{\mathbf{r}}_C + \dot{\boldsymbol{\rho}}_i \right) \cdot \left(\dot{\mathbf{r}}_C + \dot{\boldsymbol{\rho}}_i \right) \right] \Big|_{A_i}^{B_i} = \frac{1}{2} m_i \left(\dot{\mathbf{r}}_C \cdot \dot{\mathbf{r}}_C + 2 \dot{\mathbf{r}}_C \cdot \dot{\boldsymbol{\rho}}_i + \dot{\boldsymbol{\rho}}_i \cdot \dot{\boldsymbol{\rho}}_i \right) \Big|_{A_i}^{B_i} \end{aligned}$$

The total work done over all the particles is:

$$\begin{aligned} W &= \sum_{i=1}^{N} W_i = \frac{1}{2} \sum_{i=1}^{N} m_i \left(\dot{\mathbf{r}}_C \cdot \dot{\mathbf{r}}_C + 2 \dot{\mathbf{r}}_C \cdot \dot{\boldsymbol{\rho}}_i + \dot{\boldsymbol{\rho}}_i \cdot \dot{\boldsymbol{\rho}}_i \right) \Big|_{A_i}^{B_i} \\ &= \frac{1}{2} \left[\left(\dot{\mathbf{r}}_C \cdot \dot{\mathbf{r}}_C \right) \sum_{i=1}^{N} m_i + 2 \dot{\mathbf{r}}_C \cdot \sum_{i=1}^{N} m_i \dot{\boldsymbol{\rho}}_i + \sum_{i=1}^{N} m_i \dot{\boldsymbol{\rho}}_i \cdot \dot{\boldsymbol{\rho}}_i \right] \Bigg|_{A_i}^{B_i} \end{aligned} \tag{10}$$

The second term in this expression is zero because from the definition of center of mass:

$$\sum_{i=1}^{N} m_i \dot{\boldsymbol{\rho}}_i = \sum_{i=1}^{N} m_i \left(\dot{\mathbf{r}}_i - \dot{\mathbf{r}}_C \right) = \sum_{i=1}^{N} m_i \dot{\mathbf{r}}_i - \dot{\mathbf{r}}_C \sum_{i=1}^{N} m_i = \sum_{i=1}^{N} m_i \dot{\mathbf{r}}_i - m \dot{\mathbf{r}}_C = 0$$

Thus Eq. (10) becomes

$$\begin{aligned} W &= \frac{1}{2}m\left(\dot{\mathbf{r}}_C\cdot\dot{\mathbf{r}}_C\right)\Big|_{A_C}^{B_C} + \frac{1}{2}\left(\sum_{i=1}^{N} m_i\dot{\boldsymbol{\rho}}_i\cdot\dot{\boldsymbol{\rho}}_i\right)\Bigg|_{A_i}^{B_i} \\ &= \frac{1}{2}m\left(v_{CB}^2 - v_{CA}^2\right) + \frac{1}{2}\left(\sum_{i=1}^{N} m_i\dot{\boldsymbol{\rho}}_i\cdot\dot{\boldsymbol{\rho}}_i\right)\Bigg|_{A_i}^{B_i} \\ &= \left(T_{CB} - T_{CA}\right) + \frac{1}{2}\left(\sum_{i=1}^{N} m_i\dot{\boldsymbol{\rho}}_i\cdot\dot{\boldsymbol{\rho}}_i\right)\Bigg|_{A_i}^{B_i} = T_B - T_A \end{aligned} \tag{11}$$

The first term in this expression is identical to the right side of Eq. (7). The second term is the change in kinetic energy due to the motion of the particles relative to the center of mass. The sum of the two terms expresses the change in the total kinetic energy of the particles.

Eq. (11) states that the total work done by the external and internal forces acting on the particles in moving them along their individual paths equals the change in the kinetic energy the system due to the motion of the center of mass plus a change in kinetic energy due to the motion of the particles relative to the center of mass. Furthermore the two components of work in Eq. (9) and the two components of kinetic energy in Eq. (11) are separately assignable to each other. The first two components are related by Eq. (7). Thus the second two components are related by:

$$\sum_{i=1}^{N}\int_{A_i}^{B_i}\left(\mathbf{F}_i + \sum_{j=1}^{N}\mathbf{f}_{ij}\right)\cdot d\boldsymbol{\rho}_i = \frac{1}{2}\left(\sum_{i=1}^{N} m_i\dot{\boldsymbol{\rho}}_i\cdot\dot{\boldsymbol{\rho}}_i\right)\Bigg|_{A_i}^{B_i} \tag{12}$$

The result expressed by Eq. (11) is known as Koenig's Theorem.

Based on Eq. (11) the total kinetic energy of a system of particles can be written as

$$T = \frac{1}{2}m\left(\dot{\mathbf{r}}_C\cdot\dot{\mathbf{r}}_C\right) + \frac{1}{2}\left(\sum_{i=1}^{N} m_i\dot{\boldsymbol{\rho}}_i\cdot\dot{\boldsymbol{\rho}}_i\right)$$

Conservative Systems

The system of particles is said to be conservative if all the external and internal forces acting on the particles are conservative. More specifically in this case:

$$\mathbf{F}_i \cdot d\mathbf{r}_i = -dV_i \text{ and } \mathbf{f}_{ij} \cdot d\mathbf{r}_i = -dU_{ij}$$

Then an overall potential energy function can be written as:

$$dV = \sum_{i=1}^{N} \left(dV_i + \sum_{j=1}^{N} dU_{ij} \right)$$

and the principle of conservation of mechanical energy can be written as in the case of a single particle:

$$V_A + T_A = V_B + T_B \quad (13)$$

Note that if only the external forces are conservative with respect to the center of mass (as in the case of gravity) we have:

$$\mathbf{F} \cdot d\mathbf{r}_C = -dV_C$$

In this case the principle of conservation of energy can be written only with respect to the motion of the center of mass:

$$V_{CA} + T_{CA} = V_{CB} + T_{CB} \quad (14)$$

Moment-Angular Momentum Relation

Consider particle *i* whose position relative to the origin *O* of an inertial reference frame and an arbitrary point *P* is defined by the vectors $\mathbf{r}_i$ and $\boldsymbol{\rho}_i$, respectively, as shown in Fig. 3. The vectors $\mathbf{r}_C$ and $\boldsymbol{\rho}_C$ locate the center of mass relative to *O* and *P*, respectively. Point *P* is assumed to be in arbitrary motion. The other particles in the system are not shown.

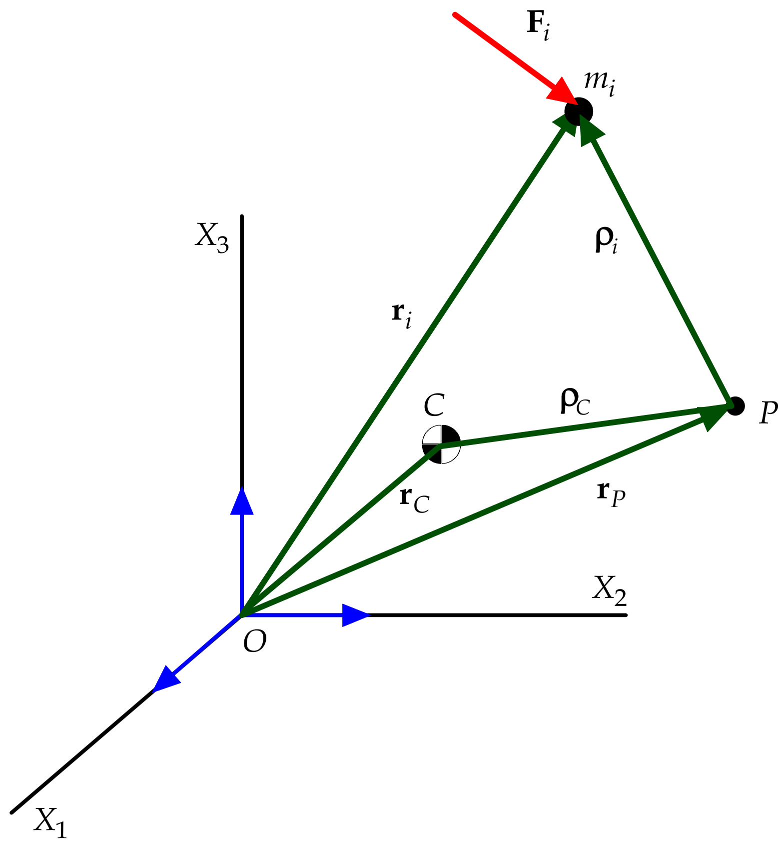

Figure 3. Location of particle *i* relative to center of mass and point *P*

From Fig. 3 it is clear that

$$\begin{aligned} \mathbf{r}_i &= \mathbf{r}_P + \boldsymbol{\rho}_i \\ \mathbf{r}_C &= \mathbf{r}_P + \boldsymbol{\rho}_C \end{aligned} \tag{15}$$

Taking the cross product of both sides of Eq. (2) with $\mathbf{r}_i$ we obtain:

$$\mathbf{r}_i \times \left(\mathbf{F}_i + \sum_{j=1}^{N} \mathbf{f}_{ij} \right) = \mathbf{r}_i \times m_i \ddot{\mathbf{r}}_i \tag{16}$$

Using the first relation in Eq. (15) and summing the left side of this equation over i we obtain:

$$\begin{aligned} &\sum_{i=1}^{N} (\mathbf{r}_P + \boldsymbol{\rho}_i) \times \left(\mathbf{F}_i + \sum_{j=1}^{N} \mathbf{f}_{ij} \right) \\ &= \mathbf{r}_P \times \sum_{j=1}^{N} \mathbf{F}_i + \mathbf{r}_P \times \sum_{i=1}^{N} \sum_{j=1}^{N} \mathbf{f}_{ij} + \sum_{i=1}^{N} (\boldsymbol{\rho}_i \times \mathbf{F}_i) + \sum_{i=1}^{N} \left(\boldsymbol{\rho}_i \times \sum_{j=1}^{N} \mathbf{f}_{ij} \right) \end{aligned} \tag{17}$$

Clearly the second and fourth terms of this expression vanish by the same arguments used for Eq. (3). Thus Eq. (17) reduces to:

$$\sum_{i=1}^{N} (\mathbf{r}_P + \boldsymbol{\rho}_i) \times \left(\mathbf{F}_i + \sum_{j=1}^{N} \mathbf{f}_{ij} \right) = \mathbf{r}_P \times \mathbf{F} + \sum_{j=1}^{N} (\boldsymbol{\rho}_i \times \mathbf{F}_i)$$

The last term in this expression is the resultant moment of all the external forces about the point P. Thus we can write:

$$\sum_{i=1}^{N} (\mathbf{r}_P + \boldsymbol{\rho}_i) \times \left(\mathbf{F}_i + \sum_{j=1}^{N} \mathbf{f}_{ij} \right) = \mathbf{r}_P \times \mathbf{F} + \mathbf{M}_P \tag{18}$$

The right side of Eq. (16) can be written as:

$$\mathbf{r}_i \times m_i \ddot{\mathbf{r}}_i = (\mathbf{r}_P + \boldsymbol{\rho}_i) \times m_i \ddot{\mathbf{r}}_i = \mathbf{r}_P \times m_i \ddot{\mathbf{r}}_i + \boldsymbol{\rho}_i \times m_i (\ddot{\mathbf{r}}_P + \ddot{\boldsymbol{\rho}}_i)$$

Summing these terms over i we obtain:

$$\sum_{i=1}^{N} \mathbf{r}_i \times m_i \ddot{\mathbf{r}}_i = \mathbf{r}_P \times \sum_{i=1}^{N} m_i \ddot{\mathbf{r}}_i + \left(\sum_{i=1}^{N} m_i \boldsymbol{\rho}_i \right) \times \ddot{\mathbf{r}}_P + \sum_{i=1}^{N} (\boldsymbol{\rho}_i \times m_i \ddot{\boldsymbol{\rho}}_i) \tag{19}$$

From Eq. (3) $\sum_{i=1}^{N} m_i \ddot{\mathbf{r}}_i = \sum_{i=1}^{N} \mathbf{F}_i = \mathbf{F}$. Also using the two relations in Eq. (15)

$$\sum_{i=1}^{N} m_i \boldsymbol{\rho}_i = \sum_{i=1}^{N} m_i (\mathbf{r}_i - \mathbf{r}_P) = \sum_{i=1}^{N} m_i \mathbf{r}_i - \left(\sum_{i=1}^{N} m_i \right) \mathbf{r}_P$$
$$= m\mathbf{r}_C - m\mathbf{r}_P = m(\mathbf{r}_C - \mathbf{r}_P) = m\boldsymbol{\rho}_C$$

Eq. (19) thus becomes:

$$\sum_{i=1}^{N} \mathbf{r}_i \times m_i \ddot{\mathbf{r}}_i = \mathbf{r}_P \times \mathbf{F} + m\boldsymbol{\rho}_C \times \ddot{\mathbf{r}}_P + \sum_{i=1}^{N} (\boldsymbol{\rho}_i \times m_i \ddot{\boldsymbol{\rho}}_i) \tag{20}$$

If the angular momentum of all the particles about P is defined as

$$\mathbf{H}_P = \sum_{i=1}^{N} (\boldsymbol{\rho}_i \times m_i \dot{\boldsymbol{\rho}}_i)$$

then it is evident that the last term in Eq. (20) is

$$\sum_{i=1}^{N} (\boldsymbol{\rho}_i \times m_i \ddot{\boldsymbol{\rho}}_i) = \dot{\mathbf{H}}_P \tag{21}$$

Finally combining Eqs. (18), (20), and (21) we obtain:

$$\mathbf{r}_P \times \mathbf{F} + \mathbf{M}_P = \mathbf{r}_P \times \mathbf{F} + m\boldsymbol{\rho}_C \times \ddot{\mathbf{r}}_P + \dot{\mathbf{H}}_P$$

or

$$\dot{\mathbf{H}}_P = \mathbf{M}_P - m\boldsymbol{\rho}_C \times \ddot{\mathbf{r}}_P \tag{22}$$

Eq. (22) states that the net resultant moment of external forces acting on a system of particles about a point in general motion equals the rate of change of the total angular momentum of the particles about the same point plus a term that compensates for the acceleration of the point.

Eq. (22) reduces to the more familiar form

$$\dot{\mathbf{H}}_P = \mathbf{M}_P \tag{23}$$

if

i) $\ddot{\mathbf{r}}_P = 0$ (i.e. if point P is fixed in an inertial reference frame);
ii) $\boldsymbol{\rho}_C = \mathbf{0}$ (i.e. if the angular momentum and the moment are taken about the center of mass)
iii) $\boldsymbol{\rho}_C \times \ddot{\mathbf{r}}_P = \mathbf{0}$ (i.e. if the two vectors are parallel or colinear)

The third case occurs in the case of the instantaneous center of rotation of a wheel rolling without slip on a surface.

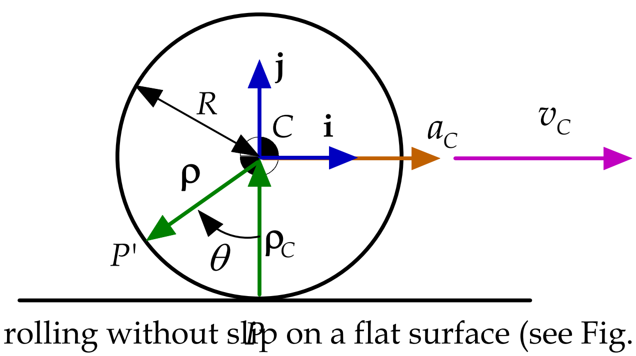

Figure 4. A wheel rolling without slip on a flat surface

Consider a wheel or radius R rolling without slip on a flat surface (see Fig. 4). The velocity and acceleration of the center of mass of the wheel are v_C and a_C to the right as shown. Using a nonrotating reference frame ($\boldsymbol{\omega}=0$) moving with *C*, the position vector locating the point *P* at some arbitrary time is given by:

$$\boldsymbol{\rho}=-R\left(\sin\theta\mathbf{i}+\cos\theta\mathbf{j}\right)$$

Thus the velocity of point *P* at some arbitrary time is:

$$\mathbf{v}_P=\mathbf{v}_C+\boldsymbol{\omega}\times\boldsymbol{\rho}+\left(\dot{\boldsymbol{\rho}}\right)_r=\mathbf{v}_C+\left(\dot{\boldsymbol{\rho}}\right)_r$$

which results in:

$$\mathbf{v}_P=v_C\mathbf{i}-R\dot{\theta}\left(\cos\theta\mathbf{i}-\sin\theta\mathbf{j}\right)$$

At the instant when *P* is in contact with the flat surface $\theta=0$ which results in

$$\mathbf{v}_P=\left(v_C-R\dot{\theta}\right)\mathbf{i}$$

The rolling with no slip condition implies that $v_C=R\dot{\theta}$. Thus the velocity of *P* at the instant it is in contact with the surface is

$$\mathbf{v}_P=0$$

as expected.

The acceleration of *P* at some arbitrary time is:

$$\mathbf{a}_P=\mathbf{a}_C+\dot{\boldsymbol{\omega}}\times\boldsymbol{\rho}+\boldsymbol{\omega}\times\left(\boldsymbol{\omega}\times\boldsymbol{\rho}\right)+\left(\ddot{\boldsymbol{\rho}}\right)_r+2\boldsymbol{\omega}\times\left(\dot{\boldsymbol{\rho}}\right)_r=\mathbf{a}_C+\left(\ddot{\boldsymbol{\rho}}\right)_r$$

which results in:

$$\mathbf{a}_P=a_C\mathbf{i}-R\ddot{\theta}\left(\cos\theta\mathbf{i}-\sin\theta\mathbf{j}\right)+R\dot{\theta}^2\left(\sin\theta\mathbf{i}+\cos\theta\mathbf{j}\right)$$

At the instant when P is in contact with the flat surface $\theta = 0$ which results in

$$\mathbf{a}_P = \left(a_C - R\ddot{\theta}\right)\mathbf{i} + R\dot{\theta}^2\mathbf{j}$$

The rolling with no slip condition implies that $a_C = R\ddot{\theta}$. Thus the acceleration of P at the instant it is in contact with the surface is

$$\mathbf{a}_P = R\dot{\theta}^2\mathbf{j} = \frac{v_C^2}{R}\mathbf{j}$$

This vector is parallel to the vector $\boldsymbol{\rho}_C$ shown in Fig. 4 and thus can be used as a point about which moment-angular momentum balance equations can be written without any compensating terms to account for the acceleration of the point.

In general it is safest to write moment-angular momentum balance equations about the center of mass.

If Eq. (23) holds and if $M_{Pi} = 0$ for any $i = 1,2,3$ $H_{Pi} = \text{constant}$.

<u>Angular Impulse-Angular Momentum Relation</u>

Eq. (23) can also be written as:

$$\mathbf{M}_P\, dt = d\mathbf{H}_P$$

Integrating both sides of this equation we obtain:

$$\int_{t_A}^{t_B} \mathbf{M}_P \, dt = \mathcal{M}_P = \int_A^B d\mathbf{H}_P = \mathbf{H}_{PB} - \mathbf{H}_{PA} \tag{24}$$

The quantity $\mathcal{M}_P$ is known as the total angular impulse of the net external resultant force **F** about P over the time interval $\Delta t = t_B - t_A$. Eq. (24) states that the total angular impulse of the net external resultant force acting on a system of particles about a point P that satisfies one of the three conditions following Eq. (23) over a time interval equals the change in the total angular momentum of the particles about the same point over the same time interval. Note that this is a vector equation, and if $M_i = 0$ for any $i = 1,2,3$ $H_{Bi} = H_{Ai}$. In such instances the angular momentum of the particle is said to have been conserved in the given direction. Note also that if P is accelerating then the correct relation to use is:

$$\dot{\mathbf{H}}_P = \int_{t_A}^{t_B} \mathbf{M}_P dt = \mathbf{H}_{PB} - \mathbf{H}_{PA} + \int_{t_A}^{t_B} \left(m\boldsymbol{\rho}_C \times \ddot{\mathbf{r}}_P \right) dt$$