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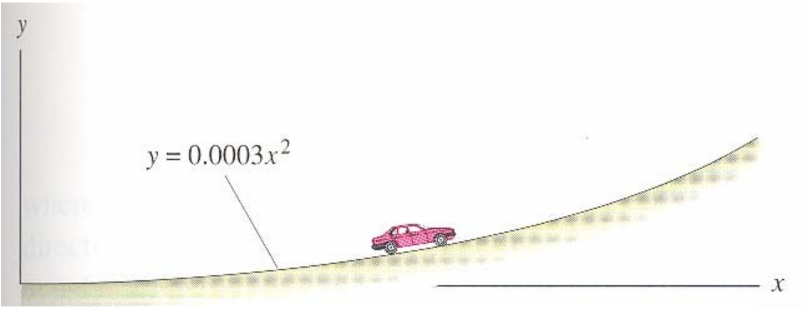
سایت آموزش مهندسی مکانیک ایران

Example 3.2

A car travels at a constant speed of 100 km/hr along a curved road of increasing grade whose vertical profile is approximated by the equation $y = 0.0003x^2$. What are the horizontal and vertical components of the acceleration of the car when its horizontal coordinate is $x = 400$ m?



Solution:

Let is first solve a general problem of this class. Let a particle travel along a curve defined by $y = y(x)$ (see Fig. 1) with constant speed v. To save space we use the following notation:

$$\frac{dy}{dt} = \dot{y} \quad \frac{d^2y}{dt^2} = \ddot{y} \quad \frac{dy}{dx} = y' \quad \frac{d^2y}{dx^2} = y'' \quad \frac{dx}{dt} = \dot{x} \quad \frac{d^2x}{dt^2} = \ddot{x}$$

The first time derivative of y is given by:

$$\dot{y} = \frac{dy}{dx}\frac{dx}{dt} = y'\dot{x}$$

The velocity vector is by definition tangent to the path of the particle (see Fig. 1). The x and y components of the velocity are given by:

$$\dot{x} = v\cos\theta$$
$$\dot{y} = v\sin\theta \qquad \text{(i)}$$

where $\theta = \arctan\left(\frac{dy}{dx}\right) = \arctan(y')$.

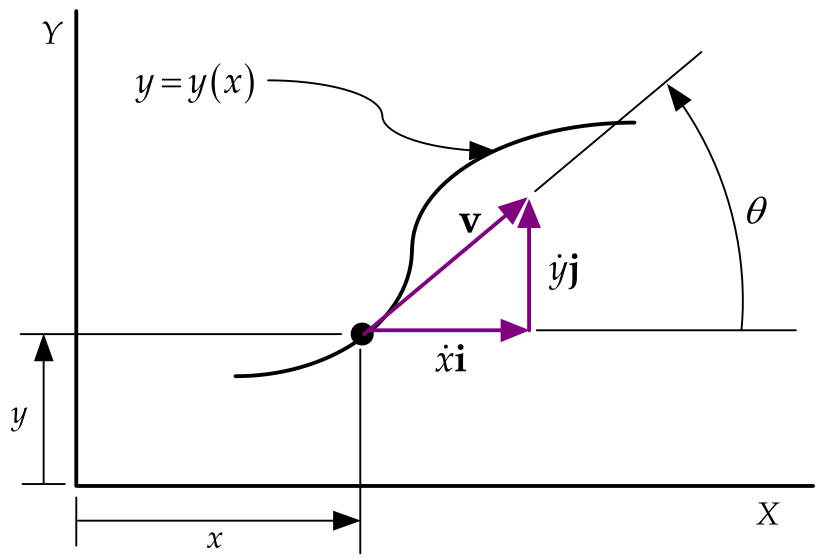


Figure 1. A particle traveling along a curve with constant speed v

Also

$$\dot{x}^2 + \dot{y}^2 = v^2 \qquad \text{(ii)}$$

and

$$\frac{\dot{y}}{\dot{x}} = \tan\theta = y'$$

The second time derivative of y (acceleration of the particle in the y direction) can be similarly obtained:

$$\ddot{y} = y''\dot{x}^2 + y'\ddot{x} \tag{iii}$$

If v is constant, differentiating (ii) with respect to time we obtain:

$$\dot{x}\ddot{x} + \dot{y}\ddot{y} = 0$$

from which we can write:

$$\ddot{y} = -\frac{\dot{x}}{\dot{y}}\ddot{x} = -\frac{\ddot{x}}{y'} \tag{iv}$$

Substituting this relation in (iii) and rearranging we obtain:

$$\left(y' + \frac{1}{y'}\right)\ddot{x} = -y''\dot{x}^2 \tag{v}$$

From (i)

$$\dot{x}^2 = v^2\cos^2\theta = \frac{v^2}{1+\tan^2\theta} = \frac{v^2}{1+y'^2}$$

Substituting this relation in (v) and rearranging we obtain:

$$\left(\frac{1+y'^2}{y'}\right)\ddot{x} = -\frac{y''}{1+y'^2}v^2$$

which gives us the horizontal acceleration of the particle in terms of the curve parameters and the constant speed of the particle:

$$\ddot{x} = -\frac{y'y''}{\left(1+y'^2\right)^2}v^2 \qquad \text{(vi)}$$

The vertical acceleration of the particle can now be obtained from (iv):

$$\ddot{y} = \frac{y''}{\left(1+y'^2\right)^2}v^2 \qquad \text{(vii)}$$

These are general relations that can be used in any problem of this class. Note that the acceleration of a particle traveling along a curve (not a straight line) is not zero even if its speed along the curve is constant. This happens because the velocity vector changes direction as the particle travels along the curve.

For the specific case of this problem $y = 0.0003x^2$, $y' = 0.0006x$, and $y'' = 0.0006$.
At $x = 400$ m these expressions evaluate to:

$$y' = 0.0024 \quad y'' = 0.0006$$

Substituting these values and the given value of v (100 km/h = 27.78 m/s) in (vi) and (vii) we obtain:

$$\ddot{x} = -0.0993 \text{ m/s}^2$$

$$\ddot{y} = 0.4139 \text{ m/s}^2$$