B
Supplementary Mathematics

B.1 VECTOR SPACE

Fundamental to the discipline of matrix theory as well as the operator theory of functional analysis is the definition of a *linear space*, also called a *vector space*. A linear space, denoted by V, is a collection of objects (vectors or functions in the cases of interest here) for which the following statements hold for all elements $\mathbf{x}$, $\mathbf{y}$, $\mathbf{z} \in V$ (this denotes that the vectors $\mathbf{x}$, $\mathbf{y}$, and $\mathbf{z}$ are all constrained in the set V) and for any real-valued scalars α and β:

1. $\mathbf{x} + \mathbf{y} \in V, \qquad \alpha\mathbf{x} \in V.$
2. $\mathbf{x} + \mathbf{y} = \mathbf{y} + \mathbf{x}.$
3. $(\mathbf{x} + \mathbf{y}) + \mathbf{z} = \mathbf{x} + (\mathbf{y} + \mathbf{z}).$
4. There exists an element $\mathbf{0} \in V$ such that $0\mathbf{x} = \mathbf{0}$.
5. There exists an element $1 \in V$ such that $1\mathbf{x} = \mathbf{x}$.
6. $\alpha(\beta\mathbf{x}) = (\alpha\beta)\mathbf{x}.$
7. $(\alpha + \beta)\mathbf{x} = \alpha\mathbf{x} + \beta\mathbf{x}.$
8. $\alpha(\mathbf{x} + \mathbf{y}) = \alpha\mathbf{x} + \alpha\mathbf{y}.$

The examples of linear spaces V used in this text are the set of real vectors of dimension n, the set of complex vectors of dimension n, and the set of functions that are square integrable in the Lebseque sense.

B.2 RANK

An extremely useful concept in matrix analysis is the idea of rank introduced in Section 3.2. Let $\mathbb{R}^{m \times n}$ denote the set of all $m \times n$ matrices with m rows and n columns. Consider the matrix $A \in \mathbb{R}^{m \times n}$. If the columns of matrix A are considered as vectors, the number of linearly independent columns is defined as the *column rank* of matrix A. Likewise, the number of linearly independent rows of matrix A is called the *row rank* of A. The row rank of a matrix and the column rank of the matrix are equal, and this integer is called the *rank* of matrix A.

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The concept of rank is useful in solving equations as well as checking stability of a system (Chapter 4) or the controllability and observability of a system (Chapter 7). Perhaps the best way to determine the rank of a matrix is to calculate the singular values of the matrix of interest (see Section 7.7 and the following comments). The rank of a matrix can be shown to be equal to the number of nonzero singular values of the matrix. The singular values also provide a very precise way of investigating the numerical difficulties frequently encountered in situations where the rank of the matrix is near the desired value. This shows up as very small but nonzero singular values, as discussed following Equation (8.53).

A simple procedure to calculate the singular values of a matrix A, and hence determine its rank, is provided by calculating the eigenvalues of the symmetric matrix:

$$\tilde{A} = \begin{bmatrix} 0 & A^T \\ A & 0 \end{bmatrix}$$

If $A \in R^{m \times n}$ of rank r, the first r eigenvalues of $\tilde{A}$ are equal to the singular values of A, the next r eigenvalues are equal to the negative of the singular values of A, and the remaining eigenvalues of $\tilde{A}$ are zero. The rank of A is thus the number of positive eigenvalues of the symmetric matrix $\tilde{A}$.

B.3 INVERSES

For $A \in \mathbb{R}^{m \times n}$ the linear equation

$$A\mathbf{x} = \mathbf{b}$$

with det $A \neq 0$ has the solution $\boldsymbol{x} = A^{-1}\mathbf{b}$, where A^{-1} denotes the unique inverse of matrix A. The matrix A^{-1} is the matrix that satisfies

$$A^{-1}A = AA^{-1} = I_n$$

Next, consider $A \in \mathbb{R}^{m \times n}$. If $m > n$ and if the rank of A is n, then there exists an $n \times m$ matrix A_{L} of rank n such that

$$A_L A = I_n$$

where I_n denotes the $n \times n$ identity matrix. The matrix A_L is called the left inverse of A. If, on the other hand, $n > m$ and the rank of A is m, then there exists an $n \times m$ matrix A_R of rank m, called a right inverse of A, such that

$$AA_R = I_m$$

Where I_m denotes the $m \times m$ identity matrix. If $m = n =$ rank A, then A is nonsingular and $A_R = A_L = A^{-1}$.

Consider the matrix $A^T A$ and note that it is an $n \times n$ symmetric matrix. If A is of rank n (this requires that $m > n$), then $A^T A$ is nonsingular. A solution of

$$A\mathbf{x} = \mathbf{b}$$

for $A \in \mathbb{R}^{m\times n}$ can then be calculated by multiplying both sides of this last expression by $(A^TA)^{-1}A^T$, which yields

$$\mathbf{x} = (A^TA)^{-1}A^T\mathbf{b}$$

The quantity $(A^TA)^{-1}A^T$ is called the *generalized inverse* of A, denoted by $A^\dagger$.

The matrix $A^\dagger$ is also called a *pseudoinverse* or *Moore–Penrose inverse* and can be expressed in terms of a singular-value decomposition (Section 7.7) of matrix A. In the notation of Section 7.7, any matrix $A \in \mathbb{R}^{m\times n}$ can be expressed in terms of its singular-value factors as

$$A = U\Sigma V^T$$

where Σ denotes the diagonal matrix of singular values of A and U and V are orthogonal. For the case where $m > n$, if the rank of A is r, then the last $n - r$ (or $m - r$ if $m < n$) singular values are zero, so that Σ has the partitioned form

$$\Sigma = \begin{bmatrix} \Sigma_r & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}$$

where the zeros indicate matrices of zeros of the appropriate size and Σ_r is an $r \times r$ diagonal matrix of the nonzero singular values of A. Define the matrix Σ' by

$$\Sigma' = \begin{bmatrix} \Sigma_r^{-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}$$

The matrix $A^\dagger$ can be shown to be

$$A^\dagger = V\Sigma U^T$$

which is the singular-value decomposition of the generalized inverse. This last expression constitutes a more numerically stable way of calculating the generalized inverse than using the definition $(A^TA^{-1})A^T$.

The following Moore–Penrose conditions can be stated for the pseudoinverse. If $A \in \mathbb{R}^{m\times n}$ has the singular-value decomposition $A = U\Sigma V^T$, then $A^\dagger = V\Sigma' U^T$ satisfies

$$AA^\dagger A = A$$

$$A^\dagger AA^\dagger = A^\dagger$$

$$(AA^\dagger)^T = AA^\dagger$$

$$(A^\dagger A)^T = A^\dagger A$$

The matrix $A^\dagger$ satisfying all four of these conditions is unique. If A has full rank, then $A^\dagger$ is identical to the left (and right) inverse just discussed.

Finally, note that the least-squares solution of the general equation $A\mathbf{x} = \mathbf{b}$ calculated by using the generalized inverse of A is *not* a solution in the sense that $\mathbf{x} = A^{-1}\mathbf{b}$ is a solution in the nonsingular case but is rather a vector $\mathbf{x}$ that minimizes the quantity $||A\mathbf{x} - \mathbf{b}||$.

The preceding is a quick summary of material contained in most modern texts on linear algebra and matrix theory, such as the excellent text by Ortega (1987). Computational issues and algorithms are discussed in the text by Golub and Van Loan (1983), which also mentions several convenient software packages. In most cases, the matrix computations required in the vibration analysis covered in this text can be performed by using standard software packages, most of which are in the public domain.

REFERENCES

Golub, G.H. and Van Loan, C.F. (1983) *Matrix Computations*, Johns Hopkins University Press, Baltimore, Maryland.

Ortega, J.M. (1987) *Matrix Theory*, Plenum Press, New York.