4
Stability

4.1 INTRODUCTION

A rough idea concerning the concept of stability was introduced for single-degree-of-freedom systems in the first chapter. It was pointed out that the sign of the coefficients of the acceleration, velocity, and displacement terms determined the stability behavior of a given single-degree-of-freedom system. That is, if the coefficients have the proper sign, the motion will always remain within a given bound. This idea is extended in this chapter to the multiple-degree-of-freedom systems described in the previous two chapters. As in the case of the oscillatory behavior discussed in Chapter 3, the criterion based on the sign of the coefficients is translated into a criterion based on the definiteness of certain coefficient matrices.

It should be noted that no universal definition of stability exists, but rather variations are adopted depending on the nature of the particular problem under consideration. However, all definitions of stability are concerned with the response of a system to certain disturbances and whether or not the response stays within certain bounds.

4.2 LYAPUNOV STABILITY

The majority of the work done on the stability behavior of dynamical systems is based on a formal definition of stability given by Lyapunov (see, for instance, Hahn, 1963). This definition is stated with reference to the equilibrium point, $\mathbf{x}_0$, of a given system. In the case of the linear systems considered in this chapter, the equilibrium point can always be taken to be the zero vector. In addition, the definition of Lyapunov is usually stated in terms of the state vector of a given system rather than in physical coordinates directly, so that the equilibrium point refers to both the position and velocity.

Let $\mathbf{x}(0)$ represent the vector of initial conditions for a given system (both position and velocity). The system is said to have a *stable equilibrium* if, for any arbitrary positive number ε, there exists some positive number $\delta(\varepsilon)$ such that, whenever $||\mathbf{x}(0)|| < \delta$, then $||\mathbf{x}(t)|| < \varepsilon$ for all values of $t > 0$. A physical interpretation of this mathematical definition is that, if the initial state is within a certain value, i.e., $||\mathbf{x}(0)|| < \delta(\varepsilon)$, then the motion stays within another bound for all time, i.e., $||\mathbf{x}(t)|| < \varepsilon$. Here, $||\mathbf{x}(t)||$, called the norm of $\mathbf{x}$, is defined by $||\mathbf{x}(t)|| = (\mathbf{x}^T\mathbf{x})^{1/2}$.

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To apply this definition to the single-degree-of-freedom system of Equation (1.1), note that $\mathbf{x}(t) = [x(t)\ \dot{x}(t)]^T$. Hence

$$||\mathbf{x}(t)|| = (\mathbf{x}^T\mathbf{x})^{1/2} = \sqrt{x^2(t) + \dot{x}^2(t)}$$

For the sake of illustration, let the initial conditions be given by $\mathbf{x}(0) = 0$ and $\dot{\mathbf{x}}(0) = \omega_n = \sqrt{k/m}$. Then the solution is given by $x(t) = \sin \omega_n t$. Intuitively, this system has a stable response as the displacement response is bounded by 1, and the velocity response is bounded by ω_n. The following simple calculation illustrates how this solution satisfies the Lyapunov definition of stability.

First, note that

$$\|\mathbf{x}(0)\| = (\mathbf{x}^2(0) + \dot{\mathbf{x}}^2(0))^{1/2} = (0 + \omega_n^2)^{1/2} = \omega_n \tag{4.1}$$

and that

$$\|\mathbf{x}(t)\| = [\sin^2 \omega_n t + \omega_n^2 \cos^2 \omega_n t]^{1/2} < (1 + \omega_n^2)^{1/2} \tag{4.2}$$

These expressions show exactly how to choose δ as a function of ε for this system. From Equation (4.2) note that, if $(1 + \omega_n^2)^{1/2} < \varepsilon$, then $\|x(t)\| < \varepsilon$. From Equation (4.1) note that, if $\delta(\varepsilon)$ is chosen to be $\delta(\varepsilon) = \varepsilon\omega_n(1 + \omega_n^2)^{-1/2}$, then the definition can be followed directly to show that, if

$$\|\mathbf{x}(0)\| = \omega_n < \delta(\varepsilon) = \frac{\varepsilon\omega_n}{\sqrt{1 + \omega_n^2}}$$

is true, then $\omega_n < \varepsilon\omega_n/\sqrt{1 + \omega_n^2}$. This last expression yields

$$\sqrt{1 + \omega_n^2} < \varepsilon$$

That is, if $||\mathbf{x}(0)|| < \delta(\varepsilon)$, then $\sqrt{1 + \omega_n^2} < \varepsilon$ must be true, and Equation (4.2) yields that

$$\|\mathbf{x}(t)\| \leq \sqrt{1 + \omega_n^2} < \varepsilon$$

Hence, by a judicious choice of the function $\delta(\varepsilon)$, it has been shown that, if $||\mathbf{x}(0)|| < \delta(\varepsilon)$, then $||\mathbf{x}(t)|| < \varepsilon$ for all $t > 0$. This is true for any arbitrary choice of the positive number ε.

The preceding argument demonstrates that the undamped harmonic oscillator has solutions that satisfy the formal definition of Lyapunov stability. If dissipation, such as viscous damping, is included in the formulation, then not only is this definition of stability satisfied, but also

$$\lim_{t\to\infty} \|\mathbf{x}(t)\| = 0 \tag{4.3}$$

Such systems are said to be *asymptotically stable*. As in the single-degree-of-freedom case, if a system is asymptotically stable it is also stable. In fact, by definition, a system is asymptotically stable if it is stable and the norm of its response goes to zero as t becomes large. This can be seen by examining the definition of a limit (see Hahn, 1963).

The procedure for calculating $\delta(\varepsilon)$ is similar to that of calculating ε and δ for limits and continuity in beginning calculus. As in the case of limits in calculus, this definition of stability does not provide the most efficient means of checking the stability of a given system. Hence, the remainder of this chapter develops methods to check the stability properties of a given system that require less effort than applying the definition directly.

There are many theories that apply to the stability of multiple-degree-of-freedom systems, some of which are discussed here. The most common method of analyzing the stability of such systems is to show the existence of a Lyapunov function for the system. A *Lyapunov function*, denoted by $V(\mathbf{x})$, is a real scalar function of the vector $\mathbf{x}(t)$, which has continuous first partial derivatives and satisfies the following two conditions:

1. $V(\mathbf{x}) > 0$ for all values of $\mathbf{x}(t) \neq 0$.
2. $\dot{V}(\mathbf{x}) < 0$ for all values of $\mathbf{x}(t) \neq 0$.

Here, $\dot{V}(\mathbf{x})$ denotes the time derivative of the function $V(\mathbf{x})$. Based on this definition of a Lyapunov function, several extremely useful stability results can be stated. The first result states that, if there exists a Lyapunov function for a given system, then that system is *stable*. If, in addition, the function $\dot{V}(\mathbf{x})$ is strictly less than zero, then the system is *asymptotically stable*. This is called the *direct*, or *second, method* of Lyapunov. It should be noted that, if a Lyapunov function cannot be found, nothing can be concluded about the stability of the system, as the Lyapunov theorems are only sufficient conditions for stability.

The stability of a system can also be characterized by the eigenvalues of the system. In fact, it can easily be shown that a given linear system is stable if and only if it has no eigenvalue with a positive real part. Furthermore, the system will be asymptotically stable if and only if all of its eigenvalues have negative real parts (no zero real parts allowed). These statements are certainly consistent with the discussion in Section 4.1. The correctness of the statements can be seen by examining the solution using the expansion theorem (modal analysis) of the previous chapter [Equation (3.68)]. The eigenvalue approach to stability has the attraction of being both necessary and sufficient. However, calculating all the eigenvalues of the state matrix of a system is not always desirable.

The preceding statements about stability are not always the easiest criteria to check. In fact, use of the eigenvalue criteria requires almost as much calculation as computing the solution of the system. The interest in developing various different stability criteria is to find conditions that (1) are easier to check than calculating the solution, (2) are stated in terms of the physical parameters of the system, and (3) can be used to help design and/or control systems to be stable. Again, these goals can be exemplified by recalling the single-degree-of-freedom case, where it was shown that the sign of the coefficients m, c, and k determine the stability behavior of the system. To this end, more convenient stability criteria are examined on the basis of the classifications of a given physical system stated in Chapter 2.

4.3 CONSERVATIVE SYSTEMS

For conservative systems of the form

$$M\ddot{\mathbf{q}} + K\mathbf{q} = \mathbf{0} \tag{4.4}$$

where M and K are symmetric, a simple stability condition results – namely if M and K are positive definite, the eigenvalues of K are all positive, and hence the eigenvalues of the system are all purely imaginary. The solutions are then all linear combinations of terms of the form $e^{\pm\omega_n jt}$, or, by invoking Euler's formula, all terms are of the form $A\sin(\omega_n t+\phi)$. Thus, from the preceding statement, the system of Equation (4.4) is stable, since both the displacement and velocity response of the system are always less than some constant (A and $\omega_n A$ respectively) for all time and for any initial conditions.

Also, note that, if K has a negative eigenvalue, then the system has a positive real exponent. In this case one mode has a temporal coefficient of the form e^{at}, $a>0$, which grows without bound, causing the system to become unstable (note that, in this case, $\delta(\varepsilon)$ cannot be found).

The condition that K be positive definite can be coupled with the determinant condition, discussed in Section 3.3, to yield inequalities in the system parameters. In turn, these inequalities can be used as design criteria. It should be pointed out that, in most mechanical systems, K will be positive definite or positive semidefinite, unless some applied or external force proportional to displacement is present. In control theory, the applied control force is often proportional to position, as indicated in Equation (2.17) and example 2.4.4.

It is instructive to note that the function $V(\mathbf{q})$ defined by (the energy in the system)

$$V(\mathbf{q})=\frac{1}{2}[\dot{\mathbf{q}}^T(t)M\dot{\mathbf{q}}(t)+\mathbf{q}^T(t)K\mathbf{q}(t)] \tag{4.5}$$

serves as a Lyapunov function for the system in Equation (4.4). To see this, note first that $V(\mathbf{q})>0$, since M and K are positive definite, and that

$$\frac{d}{dt}[V(\mathbf{q})]=\dot{\mathbf{q}}^T M\ddot{\mathbf{q}}+\dot{\mathbf{q}}^T K\mathbf{q} \tag{4.6}$$

Now, if $\mathbf{q}(t)$ is a solution of Equation (4.4), it must certainly satisfy Equation (4.4). Thus, premultiplying Equation (4.4) by $\dot{\mathbf{q}}^T$ yields

$$\dot{\mathbf{q}}^T M\ddot{\mathbf{q}}+\dot{\mathbf{q}}^T K\mathbf{q}=0 \tag{4.7}$$

This, of course, shows that $\dot{V}(\mathbf{q})=0$. Hence, $V(\mathbf{q})$ is a Lyapunov function and, by the second method of Lyapunov, the equilibrium of the system described by Equation (4.4) is stable.

In cases where K may be positive semidefinite, the motion corresponding to the zero eigenvalue of K is called a *rigid body mode* and corresponds to a translational motion. Note that in this case Equation (4.5) is not a Lyapunov function because $V(\mathbf{q})=0$ for $\mathbf{q}\neq\mathbf{0}$, corresponding to the singularity of matrix K. Since the other modes are purely imaginary, such systems may still be considered well behaved because they consist of stable oscillations superimposed on the translational motion. This is common with moving mechanical parts. This explains why the concept of stability is defined differently in different situations. For instance, in aircraft stability, some rigid body motion is desirable.

4.4 SYSTEMS WITH DAMPING

As in the single-degree-of-freedom system case, if damping is added to a stable system (4.4), the resulting system can become asymptotically stable. In particular, if M, D, and K are all symmetric and positive definite, then the system

$$M\ddot{\mathbf{q}} + D\dot{\mathbf{q}} + K\mathbf{q} = \mathbf{0} \tag{4.8}$$

is asymptotically stable. Each of the eigenvalues of Equation (4.8) can be shown to have a negative real part. Again, since the conditions of stability are stated in terms of the definiteness of the coefficient matrices, the stability condition can be directly stated in terms of inequalities involving the physical constants of the system.

To see that this system has a stable equilibrium by using the Lyapunov direct method, note that $V(\mathbf{q})$ as defined by Equation (4.5) is still a Lyapunov function for the damped system of Equation (4.8). In this case, the solution $\mathbf{q}(t)$ must satisfy

$$\dot{\mathbf{q}}^T M\ddot{\mathbf{q}} + \dot{\mathbf{q}}^T K\mathbf{q} = -(\dot{\mathbf{q}}^T D\dot{\mathbf{q}}) \tag{4.9}$$

which comes directly from Equation (4.8) by premultiplying by $\dot{\mathbf{q}}^T(t)$. This means that the time derivative of the proposed Lyapunov function, $\dot{V}(\mathbf{q})$, is given by Equation (4.9) to be

$$\frac{d}{dt}V(\mathbf{q}(t)) = -\dot{\mathbf{q}}^T D\dot{\mathbf{q}} < 0 \tag{4.10}$$

This is negative for all nonzero values of $\mathbf{q}(t)$ because matrix D is positive definite. Hence, $V(\mathbf{q})$ defined by Equation (4.5) is in fact a Lyapunov function for the system described by Equation (4.8), and the system equilibrium is stable. Furthermore, since the inequality in expression (4.10) is strict, the equilibrium of the system is asymptotically stable.

An illustration of an asymptotically stable system is given in example 2.4.4. The matrices M, D, and K are all positive definite. In addition, the solution of problem 3.10 shows that both elements of the vector $\mathbf{q}(t)$ are combinations of the functions $e^{-at}\sin\omega_n t$, $a > 0$. Hence, each element goes to zero as t increases to infinity, as the definition (4.3) indicates it should.

4.5 SEMIDEFINITE DAMPING

An interesting situation occurs when the damping matrix in Equation (4.8) is only positive semidefinite. Then the above argument for the existence of a Lyapunov function is still valid, so that the system is stable. However, it is not clear whether or not the system is asymptotically stable. There are two equivalent answers to this question of asymptotic stability for systems with a semidefinite damping matrix.

The first approach is based on the *null space* of the matrix D. The null space of matrix D is the set of all nonzero vectors $\mathbf{x}$ such that $D\mathbf{x} = \mathbf{0}$, i.e., the set of those vectors corresponding to the zero eigenvalues of matrix D. Since D is semidefinite in this situation, there exists at least one nonzero vector $\mathbf{x}$ in the null space of D. Moran (1970) showed that, if D is semidefinite in Equation (4.8), then the equilibrium of Equation (4.8) is asymptotically stable if and only if none of the eigenvectors of matrix K lies in the null space of D.

This provides a convenient, necessary, and sufficient condition for asymptotic stability of semidefinite systems, but it requires the computation of the eigenvectors for K or at least the null space of D.

Physically, this result makes sense because, if there is an eigenvector of matrix K in the null space of D, the vector also becomes an eigenvector of the system. Furthermore, this eigenvector results in a zero damping mode for the system, and hence a set of initial conditions exists that excites the system into an undecaying harmonic motion.

The second approach avoids having to solve an eigenvector problem to check for asymptotic stability. Walker and Schmitendorf (1973) showed that the system of Equation (4.8) with semidefinite damping will be asymptotically stable if and only if

$$\text{Rank}\begin{bmatrix} D \\ DK \\ DK^2 \\ \vdots \\ DK^{n-1} \end{bmatrix} = n \tag{4.11}$$

where n is the number of degrees of freedom of the system. The rank of a matrix is the number of linearly independent rows (or columns) the matrix has (see Appendix B). This type of rank condition comes from control theory considerations and is used and explained again in Chapter 7.

These two approaches are equivalent. They essentially comment on whether or not the system can be transformed into a coordinate system in which one or more modes are undamped. It is interesting to note that, if D is semidefinite and $KM^{-1}D$ is symmetric, then the system is not asymptotically stable. This results since, as pointed out in section 3.5, if $KM^{-1}D = DM^{-1}K$, then $\tilde{K}$ and $\tilde{D}$ have the same eigenvectors and the system can be decoupled. In this decoupled form there will be at least one equation with no velocity term corresponding to the zero eigenvalue of D. The solution of this equation will not go to zero with time, and hence the system cannot be asymptotically stable.

4.6 GYROSCOPIC SYSTEMS

The stability properties of gyroscopic systems provide some very interesting and unexpected results. First, consider an undamped gyroscopic system of the form

$$M\ddot{\mathbf{q}} + G\dot{\mathbf{q}} + K\mathbf{q} = \mathbf{0} \tag{4.12}$$

where M and K are both positive definite and symmetric and where G is skew-symmetric. Since the quadratic form $\dot{\mathbf{q}}^T G\dot{\mathbf{q}}$ is zero for any choice of q, the Lyapunov function for the previous system [Equation (4.5)] still works for Equation (4.12), and the equilibrium of Equation (4.12) is stable.

If matrix K in Equation (4.12) is indefinite, semidefinite, or negative definite, the system may still be stable. This is a reflection of the fact that gyroscopic forces can sometimes be used to stabilize an unstable system. A child's spinning top provides an example of such a situation. The vertical position of the top is unstable until the top is spun, providing a stabilizing gyroscopic force.

Easy-to-use conditions are not available to check if Equation (4.12) is stable when K is not positive definite. Hagedorn (1975) has been able to show that, if K is negative definite and if the matrix $4K - GM^{-1}G$ is negative definite, then the system is definitely unstable. Several authors have examined the stability of Equation (4.12) when the dimension of the system is $n = 2$. Teschner (1977) showed that, if $n = 2$, K is negative definite, and $4K - GM^{-1}G$ is positive definite, then the system is stable. Inman and Saggio (1985) showed that, if $n = 2$, K is negative definite, $\det K > 0$, and the trace of $4K - GM^{-1}G$ is positive, then the system is stable.

Huseyin, Hagedorn, and Teschner (1983) showed that, for any degree-of-freedom system, if $4K - GM^{-1}G$ is positive definite and if the matrix $(GM^{-1}K - KM^{-1}G)$ is positive semidefinite, then the system is stable. In addition, they showed that, if $GM^{-1}K = KM^{-1}G$, then the system is stable if and only if the matrix $4K - GM^{-1}G$ is positive definite. These represent precise conditions for the stability of undamped gyroscopic systems. Most of these ideas result from Lyapunov's direct method. The various results on gyroscopic systems are illustrated in example 4.6.1. Bernstein and Bhat (1995) give additional examples and summarize known stability conditions up to 1994.

Example 4.6.1

Consider a simplified model of a mass mounted on a circular, weightless rotating shaft that is also subjected to an axial compression force. This system is described by Equation (4.12) with

$$M = I, \qquad G = 2\gamma \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \qquad K = \begin{bmatrix} c_1 - \gamma^2 - \eta & 0 \\ 0 & c_2 - \gamma^2 - \eta \end{bmatrix}$$

where γ represents the angular velocity of the shaft and η the axial force. The parameters c_1 and c_2 represent the flexural stiffness in two principal directions, as illustrated in Figure 4.1.

It is instructive to consider this problem first for fixed rotational speed ($\gamma = 2$) and for $\eta = 3$. Then the relevant matrices become ($M = I$)

$$K = \begin{bmatrix} c_1 - 7 & 0 \\ 0 & c_2 - 7 \end{bmatrix}$$

$$4K - GM^{-1}G = 4\begin{bmatrix} c_1 - 3 & 0 \\ 0 & c_2 - 3 \end{bmatrix}$$

Figure 4.2 shows plots of stable and unstable choices of c_1 and c_2 using the previously mentioned theories. To obtain the various regions of stability illustrated in Figure 4.2, consider the following calculations:

1. K positive definite implies that $c_1 - 7 > 0$ and $c_2 - 7 > 0$, a region of stable operation.
2. $\det K > 0$ implies that $(c_1 - 7)(c_2 - 7) > 0$, or that $c_1 < 7, c_2 < 7$. The $\mathrm{tr}(4K - GM^{-1}G) > 0$ implies that $4[(c_1 - 3) + (c_2 - 3)] > 0$, or that $c_1 + c_2 > 6$, which again yields a region of stable operation.
3. $4K - GM^{-1}G$ negative definite implies that $c_1 < 3$ and $c_2 < 3$, a region of unstable operation.
4. $4K - GM^{-1}G$ positive definite implies that $c_1 > 3$ and $c_2 > 3$, a region of either stable or unstable operation depending on other considerations. If, in addition, the matrix

$$GM^{-1}K - KM^{-1}G = 4\begin{bmatrix} 0 & c_1 - c_2 \\ c_1 - c_2 & 0 \end{bmatrix}$$

is zero, i.e., if $c_1 = c_2$, then the system is stable. Thus, the line $c_1 = c_2$ represents a region of stable operation for $c_1 = c_2 > 3$ and unstable operation for $c_1 = c_2 < 3$.

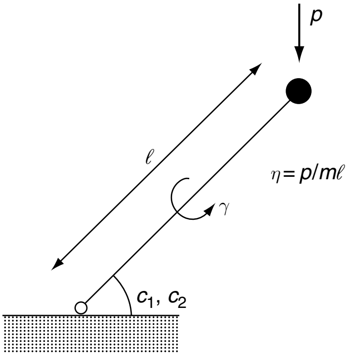

Figure 4.1 Schematic of a rotating shaft subject to an axial compression force.

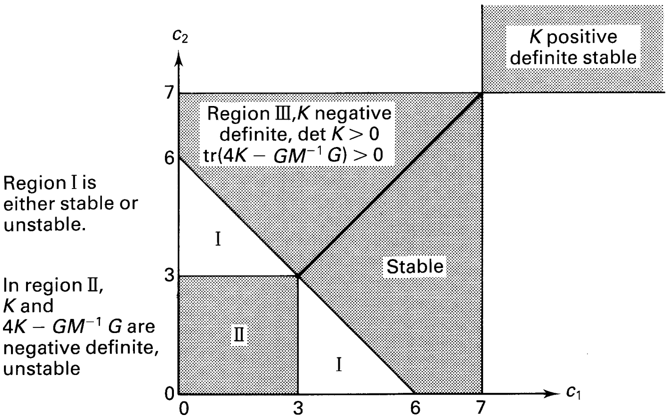

Figure 4.2 Regions of stable and unstable operation of a conservative gyroscopic system as a function of stiffness coefficients.

4.7 DAMPED GYROSCOPIC SYSTEMS

As the previous section illustrated, gyroscopic forces can be used to stabilize an unstable system. The next logical step is to consider adding damping to the system. Since added positive definite damping has caused stable symmetric systems to become asymptotically stable, the same effect is expected here. However, this turns out *not* to be the case in all circumstances.

Consider a damped gyroscopic system of the form

$$M\ddot{\mathbf{q}} + (D + G)\dot{\mathbf{q}} + K\mathbf{q} = \mathbf{0} \tag{4.13}$$

where $M = M^T > 0, D = D^T, G = -G^T$, and $K = K^T$. The following results are due to Kelvin, Tait, and Chetaev and are referred to as the KTC theorem by Zajac (1964, 1965):

1. If K and D are both positive definite, the system is asymptotically stable.
2. If K is not positive definite and D is positive definite, the system is unstable.
3. If K is positive definite and D is positive semidefinite, the system may be stable or asymptotically stable. The system is asymptotically stable if and only if none of the eigenvectors of the undamped gyroscopic system is in the null space of D. Also, proportionally damped systems will be stable.

Hughes and Gardner (1975) showed that the Walker and Schmitendorf rank condition [Equation (4.11)] also applies to gyroscopic systems with semidefinite damping and positive definite stiffness. In particular, let the state matrix A and the 'observer' matrix C be defined and denoted by

$$A = \begin{bmatrix} 0 & I \\ -M^{-1}K & -M^{-1}G \end{bmatrix}, \qquad C = [0 \quad D]$$

Then the equilibrium of Equation (4.13) is asymptotically stable if the rank of the $2n \times 2n^2$ matrix $R^T = \left[C^T \; A^T C^T \cdots (A^T)^{n-1} C^T \right]$ is full, i.e., rank $R^T = 2n$. Systems that satisfy either this rank condition or Equation (4.11) are said to be *pervasively damped*, meaning that the influence of the damping matrix D pervades each of the system coordinates. Each mode of a pervasively damped system is damped, and such systems are asymptotically stable.

Note that condition 2 points out that, if one attempts to stabilize an unstable system by adding gyroscopic forces to the system and at the same time introduces viscous damping, the system will remain unstable. A physical example of this is again given by the spinning top if the friction in the system is modeled as viscous damping. With dissipation considered, the top is in fact unstable and eventually falls over after precessing because of the effects of friction.

4.8 CIRCULATORY SYSTEMS

Next, consider those systems that have asymmetries in the coefficient of the displacement term. Such systems are called *circulatory*. A physical example is given in example 2.4.3. Other examples can be found in the fields of aeroelasticity, thermoelastic stability, and in control (see example 2.4.4). The equation of motion of such systems takes the form

$$M\ddot{\mathbf{q}} + (K + H)\mathbf{q} = \mathbf{0} \tag{4.14}$$

where $M = M^T$, $K = K^T$, and $H = -H^T$. Here, K is the symmetric part of the position coefficient and H is the skew-symmetric part. Results and stability conditions for circulatory systems are not as well developed as for symmetric conservative systems.

Since damping is not present, the stability of Equation (4.14) will be entirely determined by the matrix $A_3 = K + H$, as long as M is nonsingular. In fact, it can be shown (see Huseyin, 1978, p. 174) that Equation (4.14) is stable if and only if there exists a symmetric and positive definite matrix P such that $PM^{-1}A_3$ is symmetric and positive definite. Furthermore, if the matrix $PM^{-1}A_3$ is symmetric, there is no flutter instability. On the other hand, if such a matrix P does not exist, Equation (4.14) can be unstable both by flutter and by divergence. In this case, the system will have some complex eigenvalues with positive real parts.

The preceding results are obtained by considering an interesting subclass of circulatory systems that results from a factorization of the matrix $M^{-1}A_3$. Taussky (1972) showed that any real square matrix can be written as the product of two symmetric matrices. That is, there exist two real symmetric matrices S_1 and S_2 such that $M^{-1}A_3 = S_1S_2$. With this factorization in mind, all asymmetric matrices $M^{-1}A_3$ can be classified into two groups: those for which at least one of the matrix factors, such as S_1, is positive definite and those for which neither of the factors is positive definite. Matrices for which S_1 (or S_2) is positive definite are called *symmetrizable matrices* or *pseudosymmetric matrices*. The corresponding systems are referred to as *pseudoconservative systems*, *pseudosymmetric systems*, or *symmetrizable systems* and behave essentially like symmetric systems. One can think of this transformation as a change of coordinate systems to one in which the physical properties are easily recognized.

In fact, for $M^{-1}A_3 = S_1S_2$, S_1 positive definite, the system described by Equation (4.14) is stable if and only if S_2 is positive definite. Furthermore, if S_2 is not positive definite, instability can only occur through divergence, and no flutter instability is possible. Complete proofs of these statements can be found in Huseyin (1978), along with a detailed discussion. The proof follows from the simple idea that, if $M^{-1}A_3$ is symmetrizable, then the system is mathematically similar to a symmetric system. Thus, the stability problem is reduced to considering that of the symmetric matrix S_2.

The similarity transformation is given by the matrix $S_2^{1/2}$, the positive definite square root of matrix S_1. To see this, premultiply Equation (4.14) by $S_1^{-1/2}$, which is nonsingular. This yields

$$S_1^{-1/2}\ddot{\mathbf{q}} + S_1^{-1/2}(M^{-1}A_3)\mathbf{q} = \mathbf{0} \tag{4.15}$$

which becomes

$$S_1^{-1/2}\ddot{\mathbf{q}} + S_1^{-1/2}S_1S_2\mathbf{q} = \mathbf{0}$$

or

$$S_1^{-1/2}\ddot{\mathbf{q}} + S_1^{1/2}S_2\mathbf{q} = \mathbf{0} \tag{4.16}$$

Substitution of $\mathbf{q} = S_1^{1/2}\mathbf{y}$ into this last expression yields the equivalent symmetric system

$$\ddot{\mathbf{y}} + S_1^{1/2}S_2S_1^{1/2}\mathbf{y} = \mathbf{0} \tag{4.17}$$

Thus, there is a nonsingular transformation $S_1^{1/2}$ relating the solution of symmetric problems given by Equation (4.17) to the asymmetric problem of Equation (4.14). Because the transformation is nonsingular, the eigenvalues of Equations (4.14) and (4.17) are the same. Thus, the two representations have the same stability properties. Here, the matrix $S_1^{1/2}S_2S_1^{1/2}$ is seen to be symmetric by taking its transpose, i.e., $(S_1^{1/2}S_2S_1^{1/2})^T = S_1^{1/2}S_2S_1^{1/2}$. Thus, if S_2 is positive definite, then $S_1^{1/2}S_2S_1^{1/2}$ is positive definite (and symmetric) so that Equation (4.17) is stable. Methods for calculating the matrices S_1 and S_2 are discussed in the next section.

Note that, if the system is not symmetrizable, i.e., if S_1 is not positive definite, then $S_1^{1/2}$ does not exist and the preceding development fails. In this case, instability of Equation (4.14) can be caused by either flutter or divergence.

4.9 ASYMMETRIC SYSTEMS

For systems that have both asymmetric velocity and stiffness coefficients not falling into any of the previously mentioned classifications, several different approaches are available. The first approach discussed here follows the idea of a pseudosymmetric system introduced in the previous section, and the second approach follows methods of constructing Lyapunov functions. The systems considered in this section are of the most general form [Equation (2.7) with $\mathbf{f} = \mathbf{0}$]

$$A_1\ddot{\mathbf{q}} + A_2\dot{\mathbf{q}} + A_3\mathbf{q} = \mathbf{0} \tag{4.18}$$

where A_1 is assumed to be nonsingular, $A_2 = D + G$, and $A_3 = K + H$. Since A_1 is nonsingular and since A_2 and A_3 are symmetric, it is sufficient to consider the equivalent system

$$\ddot{\mathbf{q}} + A_1^{-1}A_2\dot{\mathbf{q}} + A_1^{-1}A_3\mathbf{q} = \mathbf{0} \tag{4.19}$$

The system described by Equation (4.19) can again be split into two classes by examining the factorization of the matrices $A_1^{-1}A_2$ and $A_1^{-1}A_3$ in a fashion similar to the previous section. First note that there exists a factorization of these matrices of the form $A_1^{-1}A_2 = T_1T_2$ and $A_1^{-1}A_3 = S_1S_2$, where the matrices S_1, S_2, T_1, and T_2 are all symmetric. This is always possible because of the result of Taussky just mentioned, i.e., any real square matrix can always be written as the product of two symmetric matrices. Then, the system in Equation (4.19) is similar to a symmetric system if and only if there exists at least one factorization of $A_1^{-1}A_2$ and $A_1^{-1}A_3$ such that $S_1 = T_1$, which is positive definite. Such systems are called *symmetrizable*. Under this assumption, it can be shown that the equilibrium position of Equation (4.18) is asymptotically stable if the eigenvalues of the matrix $A_1^{-1}A_2$ and the matrix $A_1^{-1}A_3$ are all positive real numbers. This corresponds to requiring the matrices S_2 and T_2 to be positive definite.

Deciding if the matrices $A_1^{-1}A_2$ and $A_1^{-1}A_3$ are symmetrizable is, in general, not an easy task. However, if the matrix A_2 is proportional, i.e., if $A_2 = \alpha A_1 + \beta A_3$, where α and β are scalars, and if $A_1^{-1}A_3$ is symmetrizable, then $A_1^{-1}A_2$ is also symmetrizable, and there exists a common factor S_1T_1. It can also be shown that, if two real matrices commute and one of them is symmetrizable, then the other matrix is also symmetrizable, and they can be reduced to a symmetric form simultaneously.

Several of the usual stability conditions stated for symmetric systems can now be stated for symmetrizable systems. If $A_1^{-1}A_2$ has nonnegative eigenvalues (i.e., zero is allowed) and if $A_1^{-1}A_3$ has positive eigenvalues, Equation (4.18) is asymptotically stable if and only if the $n^2 \times n$ matrix

$$R = \begin{bmatrix} A_1^{-1}A_2 \\ A_1^{-1}A_2(A_1^{-1}A_3) \\ A_1^{-1}A_2(A_1^{-1}A_3)^2 \\ \vdots \\ A_1^{-1}A_2(A_1^{-1}A_3)^{n-1} \end{bmatrix} \tag{4.20}$$

has rank n. This, of course, is equivalent to the statement made by Moran (1970) for symmetric systems, mentioned in section 4.5, that the system is asymptotically stable if and only if none of the eigenvectors of $A_1^{-1}A_3$ lies in the null space of $A_1^{-1}A_2$.

The KTC theorem can also be extended to systems with asymmetric but symmetrizable coefficients. However, the extension is somewhat more complicated. Consider the matrix $S=(A_1^{-1}A_2)T_1+T_1(A_1^{-1}A_2)^T$, and note that S is symmetric. If S is positive definite and if A_3 is nonsingular, then Equation (4.18) is stable if and only if all of the eigenvalues of $A_1^{-1}A_3$ are positive numbers. If $S_1 \neq T_1$, the matrix $A_1^{-1}A_2$ contains a gyroscopic term, and this result states the equivalent problem faced in using gyroscopic forces to stabilize an unstable system, that it cannot be done in the presence of damping. Hence, in the case where $S_1 \neq T_1$, the stability of the system is determined by the eigenvalues of T_2 (which are the eigenvalues of A_3) for systems with a symmetrizable stiffness coefficient matrix.

The following two examples serve to illustrate the above discussion as well as indicate the level of computation required.

Example 4.9.1

The preceding results are best understood by considering some examples. First, consider a system described by

$$\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\ddot{\mathbf{q}}+\begin{bmatrix}2 & 4\\ 4 & 2\end{bmatrix}\dot{\mathbf{q}}+\begin{bmatrix}10 & 8\\ 0 & 1\end{bmatrix}\mathbf{q}=\mathbf{0}$$

Here note that

$$A_1^{-1}A_2=\begin{bmatrix}2 & 4\\ 1 & 2\end{bmatrix}=\begin{bmatrix}1.2461 & -0.2769\\ -0.2769 & 0.3115\end{bmatrix}\begin{bmatrix}2.8889 & 5.7778\\ 5.7778 & 11.5556\end{bmatrix}$$

$$A_1^{-1}A_3=\begin{bmatrix}10 & 8\\ 0 & 1\end{bmatrix}=\begin{bmatrix}1.2461 & -0.2769\\ -0.2769 & 0.3115\end{bmatrix}\begin{bmatrix}10 & 8.8889\\ 8.8889 & 11.1111\end{bmatrix}$$

so that $T_1=S_1$, and the coefficient matrices have a common factor. Then the eigenvalue problem associated with this system is similar to a symmetric eigenvalue problem. An illustration on how to calculate the symmetric factors of a matrix is given in example 4.9.3.

According to the previous theorems, the stability of this equation may be indicated by calculating the eigenvalues of $A_1^{-1}A_2$ and of $A_1^{-1}A_3$. The eigenvalues of $A_1^{-1}A_2$ in this example are $\lambda_{1,2}=0, 4$, and those of $A_1^{-1}A_3$ are $\lambda_{1,2}=1, 10$. Hence, $A_1^{-1}A_3$ has positive real eigenvalues and $A_1^{-1}A_2$ has nonnegative real eigenvalues. Because of the singularity of the matrix $A_1^{-1}A_2$, knowledge of the rank of the matrix equation [Equation (4.20)] is required in order to determine if the system is asymptotically stable or just stable. The matrix of Equation (4.20) is

$$\begin{bmatrix}2 & 4\\ 1 & 2\\ 20 & 20\\ 10 & 10\end{bmatrix}\sim\begin{bmatrix}0 & 0\\ 1 & 2\\ 0 & 0\\ 1 & 1\end{bmatrix}\sim\begin{bmatrix}0 & 0\\ 1 & 0\\ 0 & 0\\ 0 & 1\end{bmatrix}$$

which obviously has rank $=2$, the value of n. Here, the symbol $\sim$ denotes column (or row) equivalence, as discussed in Appendix B. Thus, the previous result states that the equilibrium of this example is asymptotically stable. This is in agreement with the eigenvalue calculation for the system, which yields

$$\lambda_{1,2}=-1\pm 2j$$

$$\lambda_{3,4}=-1\pm j$$

showing clearly that the equilibrium is in fact asymptotically stable, as predicted by the theory.

Example 4.9.2

As a second example, consider the asymmetric problem given by

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\ddot{\mathbf{q}}+\begin{bmatrix} 9 & 20 \\ 3 & 8 \end{bmatrix}\dot{\mathbf{q}}+\begin{bmatrix} -5 & -6 \\ -4 & 0 \end{bmatrix}\mathbf{q}=\mathbf{0}$$

Premultiplying this by A_1^{-1} yields

$$I\ddot{\mathbf{q}}+\begin{bmatrix} 6 & 12 \\ 3 & 8 \end{bmatrix}\dot{\mathbf{q}}+\begin{bmatrix} -1 & -6 \\ -4 & 0 \end{bmatrix}\mathbf{q}=\mathbf{0}$$

The eigenvalues of $A_1^{-1}A_2$ are $\lambda_{1,2}=7\pm(1/2)\sqrt{148}$ and those of $A_1^{-1}A_3$ are $\lambda_{1,2}=-1/2\pm(1/2)\sqrt{97}$. Thus, both coefficient matrices have real distinct eigenvalues and are therefore symmetrizable. However, a simple computation shows that there does not exist a factorization of $A_1^{-1}A_2$ and $A_1^{-1}A_3$ such that $T_1=S_1$.

Thus, the generalized KTC theorem must be applied. Accordingly, if the matrix $(A_1^{-1}A_2)T_1+T_1(A_1^{-1}A_2)^T$ is positive definite, then the equilibrium of this system is unstable, since $A_1^{-1}A_3$ has a negative eigenvalue. To calculate T_2, note that $(A_1^{-1}A_3)=T_1T_2$, where T_1 is positive definite and hence nonsingular. Thus, multiplying by T_1^{-1} from the right results in the matrix T_2 being given by $T_2=T_1^{-1}(A_1^{-1}A_3)$.

Let T_1^{-1} be a general generic symmetric matrix denoted by

$$T_1^{-1}=\begin{bmatrix} a & b \\ b & c \end{bmatrix}$$

where it is desired to calculate a, b, and c so that T_1 is positive definite. Thus

$$T_2=\begin{bmatrix} a & b \\ b & c \end{bmatrix}\begin{bmatrix} -1 & -6 \\ -4 & 0 \end{bmatrix}=\begin{bmatrix} -a-4b & -6a \\ -b-4c & -6b \end{bmatrix}$$

Requiring T_2 to be symmetric and T_1 to be positive definite yields the following relationships for a, b, and c:

$$ac>b^2$$

$$6a=b+4c$$

This set of equations has multiple solutions; one convenient solution is $a=2$, $b=0$, and $c=3$. Then T_1 becomes

$$T_1=\begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{3} \end{bmatrix}$$

Thus, $(A_1^{-1}A_2)T_1+T_1(A_1^{-1}A_2)^T$ becomes

$$(A_1^{-1}A_2)T_1+T_1(A_1^{-1}A_2)^T=\begin{bmatrix} 6 & \frac{11}{2} \\ \frac{11}{2} & \frac{16}{3} \end{bmatrix}$$

which is positive definite. Thus, the equilibrium must be unstable.

This analysis again agrees with calculation of the eigenvalues, which are $\lambda_1=0.3742$, $\lambda_2=-13.5133$, and $\lambda_{3,4}=-0.4305\pm0.2136j$, indicating an unstable equilibrium, as predicted.

Example 4.9.3

The question of how to calculate the factors of a symmetrizable matrix is discussed by Huseyin (1978) and Ahmadian and Chou (1987). Here it is shown that the matrices $A_1^{-1}A_3 = S_1S_2$ and $A_1^{-1}A_2 = T_1T_2$ of example 4.9.2 do not have any common factorization such that $T_1 = S_1$. Hence, $A_1^{-1}A_2$ and $A_1^{-1}A_3$ cannot be simultaneously symmetrized by the same transformation.

It is desired to find a symmetric positive definite matrix P such that $PA_1^{-1}A_2$ and $PA_1^{-1}A_3$ are both symmetric. To that end, let

$$P = \begin{bmatrix} a & b \\ b & c \end{bmatrix}$$

Then

$$PA_1^{-1}A_2 = \begin{bmatrix} 6a+3b & 12a+8b \\ 6b+3c & 12b+8c \end{bmatrix}$$

and

$$PA_1^{-1}A_3 = \begin{bmatrix} -a-4b & -6a \\ -b-4c & -6b \end{bmatrix}$$

Symmetry of both matrices then requires that

$$\begin{aligned} 6b+3c &= 12a+8b \\ b+4c &= 6a \end{aligned} \tag{4.21}$$

Positive definiteness of P requires

$$\begin{aligned} a &> 0 \\ ac &> b^2 \end{aligned} \tag{4.22}$$

It will be shown that the problem posed by Equations (4.21) and (4.22) does not have a solution. Equations (4.21) may be written in matrix form as

$$\begin{bmatrix} -2 & 3 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} b \\ c \end{bmatrix} = 6a \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

which has the unique solution

$$\begin{bmatrix} b \\ c \end{bmatrix} = a \begin{bmatrix} 2.73 \\ 2.18 \end{bmatrix}$$

for all values of a. Thus, $b = 2.73a$ and $c = 2.18a$, so that $b^2 = 7.45a^2$ and $ac = 2.18a^2$. Then

$$ac = 2.18a^2 < 7.45a^2 = b^2$$

and condition (4.22) cannot be satisfied.

Other alternatives exist for analyzing the stability of asymmetric systems. Walker (1970), approaching the problem by looking for Lyapunov functions, was able to state several results for the stability of Equation (4.18) in terms of a fourth matrix R. If there exists a symmetric positive definite matrix R such that $RA_1^{-1}A_3$ is symmetric and positive definite (this is the same as requiring $A_1^{-1}A_3$ to be symmetrizable), then the system is stable if the symmetric part of $RA_1^{-1}A_2$ is positive semidefinite and asymptotically stable if the symmetric part of $RA_1^{-1}A_2$ is strictly positive definite. This result is slightly more general than the symmetrizable results just stated in that it allows the equivalent symmetric systems to have gyroscopic forces.

In addition to these results, Walker (1970) showed that, if there exists a symmetric matrix R such that $RA_1^{-1}A_2$ is skew-symmetric and $RA_1^{-1}A_3$ is symmetric, and such that R and $RA_1^{-1}A_3$ have the same definiteness, then the system is stable but not asymptotically stable.

Another approach to the stability of Equation (4.18), not depending on symmetrizable coefficients, has been given by Mingori (1970). He showed that, if the coefficient matrices $M, D, G, H,$ and K satisfy the commutivity conditions

$$\begin{aligned} HD^{-1}M &= MD^{-1}H \\ HD^{-1}G &= GD^{-1}H \\ HD^{-1}K &= KD^{-1}H \end{aligned}$$

then the stability of the system is determined by the matrix

$$Q = HD^{-1}MD^{-1}H - GD^{-1}H + K$$

This theory states that the system is stable, asymptotically stable, or unstable if the matrix Q possesses nonnegative, positive, or at least one negative eigenvalue respectively. Although the problem addressed is general, the restrictions are severe. For instance, this method cannot be used for systems with semidefinite damping (D^{-1} does not exist).

Other more complicated and more general stability conditions are due to Walker (1974) and an extension of his work by Ahmadian and Inman (1986). The methods are developed by using Lyapunov functions to derive stability and instability conditions on the basis of the direct method. These are stated in terms of the symmetry and definiteness of certain matrices consisting of various combinations of the matrices A_1, A_2, and A_3. These conditions offer a variety of relationships among the physical parameters of the system, which can aid in designing a stable or asymptotically stable system.

4.10 FEEDBACK SYSTEMS

One of the major reasons for using feedback control is to stabilize the system response. However, most structures are inherently stable to begin with, and control is applied to improve performance. Unfortunately, the introduction of active control can effectively destroy the symmetry and definiteness of the system, introducing the possibility of instability. Thus, checking the stability of a system after a control is designed is an important step. A majority of the work in control takes place in state space (first-order form). However, it is interesting to treat the control problem specifically in ‘mechanical’ or physical coordinates in order to take advantage of the natural symmetries and definiteness in the system. Lin (1981) developed a theory for closed-loop asymptotic stability for mechanical structures being controlled by velocity and position feedback. The systems considered here have the form (see Section 2.3)

$$M\ddot{\mathbf{q}} + A_2\dot{\mathbf{q}} + K\mathbf{q} = \mathbf{f}_f + \mathbf{f} \tag{4.23}$$

where $M = M^T$ is positive definite, $A_2 = D + G$ is asymmetric, and K is symmetric. Here, the vector $\mathbf{f}$ represents external disturbance forces (taken to be zero in this section) and the vector $\mathbf{f}_f$ represents the control force derived from the action of r force actuators represented by

$$\mathbf{f}_f = B_f\mathbf{u} \tag{4.24}$$

where the $r \times 1$ vector $\mathbf{u}$ denotes the r inputs, one for each control device (actuator), and B_f denotes the $n \times r$ matrix of weighting factors (influence coefficients or actuator gains) with structure determined by the actuator locations. In order to be able to feed back the position and velocity, let $\mathbf{y}$ be an $s \times 1$ vector of sensor outputs denoted and defined by

$$\mathbf{y} = C_p\mathbf{q} + C_v\dot{\mathbf{q}} \tag{4.25}$$

Here, C_p and C_v are $s \times n$ matrices of displacement and velocity influence coefficients respectively, with structure determined by the sensor locations and where s is the number of sensors. Equation (4.25) represents those coordinates that are measured as part of the control system and is a mathematical model of the transducer and signal processing used to measure the system response. The input vector $\boldsymbol{u}$ is chosen to be of the special form

$$\mathbf{u}(t) = -G_f\mathbf{y} = -G_fC_p\mathbf{q} - G_fC_v\dot{\mathbf{q}} \tag{4.26}$$

where the $r \times s$ matrix G_f consists of constant feedback gains. This form of control law is called *output feedback*, because the input is proportional to the measured output or response, $\mathbf{y}$.

In Equation (4.24) the matrix B_f reflects the location on the structure of any actuator or device being used to supply the forces $\mathbf{u}$. For instance, if an electromechanical or piezoelectric actuator is attached to mass m_1 in Figure 2.4, and if it supplies a force of the form $F_0 \sin\omega t$, the vector $\mathbf{u}$ reduces to the scalar $u = F_0\sin(\omega t)$ and the matrix B_f reduces to a vector $B_f^T = [1 \quad 0]$. Alternatively, the control force can be written as a column vector $\mathbf{u}$, and B_f can be written as a matrix

$$\mathbf{u} = \begin{bmatrix} F_0 \sin\omega t \\ 0 \end{bmatrix} \quad \text{and} \quad B_f = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

If, on the other hand, there are two actuators, one attached to m_1 supplying a force $F_1\sin(\omega_1 t)$ and one at m_2 supplying a force $F_2\sin(\omega_2 t)$, then the vector $\mathbf{u}$ becomes $\mathbf{u}^T = [F_1\sin(\omega_1 t) \quad F_2\sin(\omega_2 t)]$ and the matrix B_f becomes $B_f = I$, the 2×2 identity matrix. Likewise, if the positions x_1 and x_2 are measured, the matrices in Equation (4.25) become $C_\text{p} = I$ and $C_\text{v} = 0$, the 2×2 matrix of zeros. If only the position x_1 is measured and the control force is applied to x_2, then

$$B_f = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad C_p = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

Making the appropriate substitutions in the preceding equations and assuming no external disturbance (i.e., $\mathbf{f}=\mathbf{0}$) yields an equivalent homogeneous system, which includes the effect of the controls. It has the form

$$M\ddot{\mathbf{q}}+(D+G+B_fG_fC_v)\dot{\mathbf{q}}+(K+B_fG_fC_p)\mathbf{q}=0 \tag{4.27}$$

For the sake of notation, define the matrices $D^*=B_fG_fC_v$ and $K^*=B_fG_fC_p$. Note that, since the number of actuators r is usually much smaller than the number of modeled degrees of freedom n (the dimension of the system), the matrices K^* and D^* are usually singular. Since, in general, $D+D^*$ and $K+K^*$ may not be symmetric or positive definite, it is desired to establish constraints on any proposed control law to ensure the symmetry and definiteness of the coefficient matrices and hence the stability of the system (see problem 4.11). These constraints stem from requiring $D+D^*$ and $K+K^*$ to be symmetric positive definite. The problem of interest in control theory is how to choose the matrix G_f so that the response $\mathbf{q}$ has some desired property (performance and stability). Interest in this section focuses on finding constraints on the elements of G_f so that the response $\mathbf{q}$ is asymptotically stable or at least stable. The stability methods of this chapter can be applied to Equation (4.27) to develop these constraints. Note that the matrices $B_fG_fC_p$ and $B_fG_fC_v$ are represented as the matrices K_p and K_v respectively in Equation (2.17).

Collocated control refers to the case where the sensors are located at the same physical location as the actuators. If the sensors or the actuators add no additional dynamics, then collocated controllers provide improved stability of the closed-loop system. As seen above, the closed-loop system coefficients D^* and K^* generally lose their symmetry for many choices of B_f, C_f, B_v, and C_v. If, however, the gain matrices G_f and G_v are symmetric, and if $B_f^T=C_f$ and $B_v^T=C_v$, then the matrices D^* and K^* remain symmetric. The symmetry then results in the possibility of choosing the gain matrices so that D^* and K^* remain positive definite, ensuring closed-loop stability for stable open-loop systems (D and K positive definite). Placing sensors and actuators at the same location causes $B_f^T=C_f$ and $B_v^T=C_v$, so that collocated control enhances closed-loop stability. The controller design consists of choosing gains G_f and G_v that are symmetric and positive definite (or at least semidefinite) with collocated sensors and actuators to ensure a stable closed-loop response.

Example 4.10.1

Consider the two-degree-of-freedom system in figure 2.4, with a control force applied to m_1 and a measurement made of x_2 so that the control system is not collocated. Then the input matrix, output matrix, and symmetric control gain matrix are

$$B_f=\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix},\qquad C_p=\begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix},\qquad G=\begin{bmatrix}g_1 & 0\\ 0 & g_2\end{bmatrix}$$

Note that this is not collocated because $B_f^T\neq C_f$. The closed-loop system of Equation (4.27) becomes

$$\begin{bmatrix}m_1 & 0\\ 0 & m_2\end{bmatrix}\begin{bmatrix}\ddot{x}_1\\ \ddot{x}_2\end{bmatrix}+\begin{bmatrix}c_1+c_2 & -c_2\\ -c_2 & c_2\end{bmatrix}\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix}+\begin{bmatrix}k_1+k_2 & -k_2+g_1\\ -k_2 & k_2\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}=\begin{bmatrix}0\\ 0\end{bmatrix}$$

This has an asymmetric displacement coefficient, implying the potential loss of stability. If, on the other hand, a control force is applied to m_1 and a measurement made of x_1, then the control system is collocated and the input matrix, output matrix, and control gain matrix are

$$B_f = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \qquad C_p = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \qquad G = \begin{bmatrix} g_1 & 0 \\ 0 & g_2 \end{bmatrix}$$

so that $B_f^T = C_f$ and the closed-loop system of Equation (4.27) becomes

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} + \begin{bmatrix} c_1 + c_2 & -c_2 \\ -c_2 & c_2 \end{bmatrix} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} + \begin{bmatrix} k_1 + k_2 + g_1 & -k_2 \\ -k_2 & k_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

which is symmetric and stable for any choice of g_1 such that $k_1 + k_2 + g_1 > 0$.

The topic of control and Equation (4.27) is discussed in more detail in section 6.6 and in Chapter 7. Historically, most of the theory developed in the literature for the control of systems has been done using a state-space model of the structure. The next section considers the stability of systems in the state variable coordinate system.

4.11 STABILITY IN STATE SPACE

In general, if none of the stability results just mentioned is applicable, the problem can be cast in first-order form as given in Section 2.3. The system of Equation (4.18) then has the form

$$\dot{\mathbf{x}} = A\mathbf{x} \tag{4.28}$$

where A is a $2n \times 2n$ state matrix and $\mathbf{x}$ is a $2n$ state vector. In this setting, it can easily be shown that the system is asymptotically stable if all the eigenvalues of A have negative real parts and is unstable if A has one or more eigenvalues with positive real parts.

The search for stability by finding a Lyapunov function in first-order form leads to the *Lyapunov equation*

$$A^T B + BA = -C \tag{4.29}$$

where C is positive semidefinite and B is the symmetric, positive definite, unknown matrix of the desired (scalar) Lyapunov function:

$$V(\mathbf{x}) = \mathbf{x}^T B \mathbf{x} \tag{4.30}$$

Do not confuse the arbitrary matrices B and C used here with the B and C used for input and output matrices. To see that $V(\mathbf{x})$ is, in fact, the desired Lyapunov function, note that differentiation of Equation (4.30) yields

$$\frac{d}{dt}[V(\mathbf{x})] = \dot{\mathbf{x}}^T B \mathbf{x} + \mathbf{x}^T B \dot{\mathbf{x}} \tag{4.31}$$

Substitution of the state equation [Equation (4.28)] into Equation (4.31) yields

$$\begin{aligned} \frac{d}{dt}[V(\mathbf{x})] &= \mathbf{x}^T A^T B\mathbf{x} + \mathbf{x}^T BA\mathbf{x} \\ &= \mathbf{x}^T (A^T B + BA)\mathbf{x} \\ &= -\mathbf{x}^T C\mathbf{x} \end{aligned} \tag{4.32}$$

Here, taking the transpose of Equation (4.28) yields $\dot{\mathbf{x}}^T = \mathbf{x}^T A^T$, which is used to remove the time derivative in the second term. Hence, if $V(\mathbf{x})$ is to be a Lyapunov function, matrix C must be positive semidefinite. The problem of showing stability by this method for a system represented by matrix A then becomes one, given the symmetric positive semidefinite matrix C, of finding a positive definite matrix B such that Equation (4.29) is satisfied. This approach involves solving a system of linear equations for the $n(n+1)/2$ elements b_{ik} of matrix B.

As explained by Walker (1974), Hahn (1963) has shown that, for a given choice of symmetric positive definite matrix C, there exists a unique solution, i.e., there exists a symmetric matrix B satisfying Equation (4.29) if the eigenvalues of A, λ_i, satisfy $\lambda_i + \lambda_k \neq 0$ for all $i, k = 1, 2, \ldots, 2n$. Furthermore, matrix B is positive definite if and only if each eigenvalue of A has a negative real part, in which case the system is asymptotically stable. Matrix B is indefinite if and only if at least one eigenvalue of A has a positive real part, in which case the equilibrium of the system is unstable. Many theoretical and numerical calculations in stability theory are based on the solution of Equation (4.29). Walker (1974) has shown that this system of linear equations has a unique solution.

Solving for the eigenvalues of Equation (4.28) can involve writing out the characteristic equation of the system. In such cases where this can be done analytically and the coefficients of the characteristic equation are available, a simple stability condition exists, namely if the characteristic equation is written in the form

$$\lambda^n + a_1\lambda^{n-1} + a_2\lambda^{n-2} + \cdots + a_n = 0 \tag{4.33}$$

then the system is asymptotically stable if and only if the principal minors of the $n \times n$ *Hurwitz matrix* defined by

$$\begin{bmatrix} a_1 & 1 & 0 & 0 & \cdots & & & 0 \\ a_3 & a_2 & a_1 & 1 & \cdots & & & 0 \\ a_5 & a_4 & a_3 & a_2 & a_1 & 1 & \cdots & 0 \\ \cdot & & & & & & & \cdot \\ \cdot & & & & & & & \cdot \\ \cdot & & & & & & & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & a_n \end{bmatrix}$$

are all positive. In addition, if any of the coefficients a_i are nonpositive (i.e., negative or zero), then the system may be unstable. This is called the *Hurwitz test*.

Writing out the (determinant) principal minors of the Hurwitz matrix yields nonlinear inequalities in the coefficients that provide relationships in the physical parameters of the system. If these inequalities are satisfied, asymptotic stability is ensured.

Example 4.11.1

As an illustration of the Hurwitz method, consider determining the asymptotic stability of a system with the characteristic equation

$$\lambda^3 + a_1\lambda^2 + a_2\lambda + a_3 = 0$$

The Hurwitz matrix is

$$\begin{bmatrix} a_1 & 1 & 0 \\ a_3 & a_2 & a_1 \\ 0 & 0 & a_3 \end{bmatrix}$$

From the Hurwitz test, $a_1 > 0, a_2 > 0$, and $a_3 > 0$ must be satisfied. From the principal minors of the Hurwitz matrix, the inequalities

$$a_1 > 0$$

$$a_1 a_2 - a_3 > 0$$

$$a_1(a_2 a_3) - a_3^2 = a_1 a_2 a_3 - a_3^2 > 0$$

must be satisfied. The above set reduces to the conditions that $a_1 > 0$, $a_2 > 0$, $a_3 > 0$, and $a_1 a_2 - a_3 > 0$ be satisfied for the system to be asymptotically stable.

4.12 STABILITY BOUNDARIES

An alternative way of looking at stability has been summarized by Huseyin (1978) and involves examining the characteristic equation as a surface from which stability properties can be deduced by plotting various stability boundaries. These methods are especially useful when examining stability questions that arise because of an applied load. A typical example is the case of a circulatory force given in example 4.6.1 above.

The point of view taken here is that a system without an applied load is represented by a symmetric system. For example

$$M\ddot{\mathbf{q}} + K\mathbf{q} = \mathbf{0} \tag{4.34}$$

where M and K are positive definite and symmetric, i.e., the system is stable. This system is now subjected to a load proportional to position and results in the equation

$$M\ddot{\mathbf{q}} + (K - \eta E)\mathbf{q} = \mathbf{0} \tag{4.35}$$

where η is a parameter characterizing the magnitude of the applied load and E is a matrix representing the point, or points, of application of the load. If there are several loads present, they can be indexed, $\eta_k E_k$, summed, and included in the equation of motion as

$$M\ddot{\mathbf{q}} + (K - \sum \eta_k E_k)\mathbf{q} = \mathbf{0} \tag{4.36}$$

In some sense, this equation is similar to the feedback control systems described in the previous sections, the difference being that the extra term $\eta_k E_k$ results in this case from some physical loading of a structure, whereas the position feedback term $G_f C_p$ in Equation (4.26) results from adding actuators to the structure. In the case of feedback, the matrix $G_f C_p$ is found that causes the system to have a desired (stable) response. In the case of Equation (4.35), it is desired to see how the stability properties are affected by changes in the scalar parameter η_k.

The characteristic equation associated with Equation (4.36) is an equation (if no damping or gyroscopic terms are present) in λ^2 and the variable η_k. This is denoted by $\Delta(\lambda^2, \eta_k)$ and defined by

$$\Delta(\lambda^2, \eta_k) = \det(M\lambda^2 + K - \eta_k E_k) = 0 \tag{4.37}$$

In most circumstances, $\eta_k = 0$ corresponds to a stable state. Then the problem is to find values of η_k at which the system loses stability. The initially stable system may, in general, lose stability by either divergence or flutter. Many of the investigations using this method focus on determining which way stability is lost. The locus of points in the η_k space corresponding to zero roots, or divergence (recall Section 1.7), is called the *divergence boundary*. On the other hand, flutter instability corresponds to repeated roots with degenerate eigenvectors. The locus of points corresponding to repeated roots generates the *flutter boundary*. Together, these two curves comprise the *stability boundary*.

Huseyin (1978) showed that the flutter condition results from those values of η such that

$$\frac{\partial \Delta}{\partial \lambda^2} = 0 \tag{4.38}$$

A majority of Huseyin's text is devoted to various ways of computing stability boundaries for various classifications of systems. These curves allow design work to be done by examining the relationship of η to the stability of the system.

CHAPTER NOTES

The classification of systems in this chapter is motivated by the text of Huseyin (1978). This text provides a complete list of references for each type of system mentioned here, with the exception of the material on control systems. In addition, Huseyin's text provides an in-depth discussion of each topic. The material of Section 4.2 is standard Lyapunov (also spelled Liapunov in older literature) stability theory, and the definitions are available in most texts. The reader who understands limits and continuity from elementary calculus should be able to make the connection to the definition of stability. The material in Sections 4.3 and 4.4 is also standard fare and can be found in most texts considering stability of mechanical systems. The material of Section 4.5 on semidefinite damping results from several papers (as referenced) and is not usually found in text form. The material on gyroscopic systems presented in Section 4.6 is also from several papers. The material on damped gyroscopic systems is interesting because it violates instinct by illustrating that adding damping to a structure may not always make it 'more stable.' The next section deals with asymmetric but symmetrizable systems. The material of Section 4.9 is taken from Inman (1983). The major contributors to the theories (Walker and Huseyin) developed separate methods, which turned out to be quite similar and, in fact, related. The paper by Ahmadian and Chou (1987) should be consulted for methods of calculating symmetric factors of a matrix. Feedback control systems are presented in this chapter (Section 4.10) before they are formally introduced in Chapter 7 to drive home the fact that the introduction of a control to a system adds energy to it and can make it unstable. The topic of Section 4.11 also presents material from control texts. It is important to note that the controls community generally thinks of a stable system as one that is defined as asymptotically stable in this text, i.e., one with eigenvalues with negative real parts. Systems are said to be *marginally stable* by the controls community if the eigenvalues are all purely imaginary. This is called stable in this and other vibration texts. Recent survey articles on the stability of second-order systems are provided by Bernstein and Bhat (1995) and Nicholson and Lin (1996). Vidyasagar (2002) includes detailed stability analysis for both linear and nonlinear systems, as well as definitions of other types of stability.

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PROBLEMS

4.1 Consider the system in figure 2.4 with $c_1 = c$, $c_2 = 0$, $f = 0$, $k_1 = k_2 = k$, and $m_1 = m_2 = m$. The equation of motion is

$$m\ell\ddot{\mathbf{x}} + \begin{bmatrix} c & 0 \\ 0 & 0 \end{bmatrix}\dot{\mathbf{x}} + \begin{bmatrix} 2k & -k \\ -k & k \end{bmatrix}\mathbf{x} = \mathbf{0}$$

Use Moran's theorem to see if this system is asymptotically stable.

4.2 Repeat problem 4.1 by using Walker and Schmitendorf's theorem.

4.3 Again, consider the system in Figure 2.4, this time with $c_1 = 0$, $c_2 = c$, $f = 0$, $k_1 = k_2 = k$, and $m_1 = m_2 = m$. The equation of motion is

$$m\ell\ddot{\mathbf{x}} + \begin{bmatrix} c & -c \\ -c & c \end{bmatrix}\dot{\mathbf{x}} + \begin{bmatrix} 2k & -k \\ -k & k \end{bmatrix}\mathbf{x} = \mathbf{0}$$

Is this system asymptotically stable? Use any method.

4.4 Discuss the stability of the system of Equation (2.24) using any method. Note that your answer should depend on the relative values of η, m, E, I, and ℓ.

4.5 Calculate a Lyapunov function for the system of example 4.9.1.

4.6 Show that, for a system with symmetric coefficients, if D is positive semidefinite and $DM^{-1}K = KM^{-1}D$, then the system is *not* asymptotically stable.

4.7 Calculate the matrices and vectors B_f, $\mathbf{u}$, C_p and C_v, defined in Section 4.10 for the system in Figure 2.4 for the case where the velocities of m_1 and m_2 are measured and the actuator (f_2) at m_2 supplies a force of $-g_1\dot{\mathbf{x}}_1$. Discuss the stability of this closed-loop system as the gain g_1 is changed.

4.8 The characteristic equation of a given system is

$$\lambda^4 + 10\lambda^3 + \lambda^2 + 15\lambda + 3 = 0$$

Is this system asymptotically stable or unstable? Use a root solver to check your answer.

4.9 Consider the system defined by

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}\ddot{\mathbf{q}} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}\dot{\mathbf{q}} + \begin{bmatrix} k_1 & -k_3 \\ k_3 & k_2 \end{bmatrix}\mathbf{q} = \mathbf{0}$$

Assume a value of matrix R from the theory of Walker (1970) of the form

$$R = \begin{bmatrix} m_1 & g \\ g & m_2 \end{bmatrix}$$

and calculate relationships between the parameters m_i, k_i, and g that guarantee the stability of the system. Can the system be asymptotically stable?

4.10 Let

$$A_2 = \begin{bmatrix} c_1 & c_4 \\ c_3 & c_2 \end{bmatrix}$$

in problem 4.9 and repeat the analysis.

4.11 Let $\mathbf{y} = C_p B_f^T \mathbf{q} + C_v B_f^T \dot{\mathbf{q}}$, where C_p and C_v are restricted to be symmetric and show that the resulting closed-loop system with $G_f = I$ and $G = 0$ in Equation (4.27) has symmetric coefficients (Junkins, 1986).

4.12 For the system of problem 4.10, choose feedback matrices B_f, G_f, C_p, and C_v that make the system symmetric and stable (see problem 4.11 for a hint).

4.13 Prove that collocated control is stable for the system of problem 4.10.

4.14 The characteristic equation of a two-link structure with stiffness at each joint and loaded at the end by p_2 and at the joint by p_1 is

$$2\lambda^4 + p_2^2 + \lambda^2 p_1 + 4\lambda^2 p_2 + p_1 p_2 - 8\lambda^2 - p_1 - 4p_2 + 2 = 0$$

where the parameters of the structure are all taken to be unity (see Huseyin, 1978, p. 84). Calculate and sketch the divergence boundary in the $p_1 - p_2$ space. Discuss the flutter condition.

4.15 Use the Hurwitz test to discuss the stability of the system in problem 4.14.

4.16 Consider the system of example 2.4.4 [Equation (2.26)] and compute the B_f, G_f and C_p matrices that correspond to the control law suggested in the example.

4.17 Consider the system of example 4.10.1 with $M = I, C = 0.1K$, and $k_1 = k_2 = 2$. Compute the values for the gain g_1 that make the closed loop stable for the collocated case.

4.18 Consider the system of example 4.10.1 with $M = I, C = 0.1K$, and $k_1 = k_2 = 2$. Compute the values for the gain g_1 that make the closed loop stable for the noncollocated case.

4.19 A common way of improving the response of a system is to add damping via velocity feedback control. Consider the standard two-degree-of-freedom system in Figure 2.4 with open-loop values of $m_1 = m_2 = 1, c_1 = c_2 = 0.1, k_1 = 4$, and $k_2 = 1$. Add damping to the system using a control system that measures $\dot{x}_1$ and applies a force to the mass m_2 that is proportional to $\dot{x}_1$ (i.e., $-g\dot{x}_1$). Determine values of g that make the closed-loop response asymptotically stable.

4.20 Consider the system of problem 4.5 and calculate the eigenvalues of the state matrix in MATLAB to determine the stability.