5

Forced Response of Lumped-parameter Systems

5.1 INTRODUCTION

Up to this point, with the exception of Section 1.4 and some brief comments about feedback control, only the free response of a system has been discussed. In this chapter the forced response of a system is considered in detail. Such systems are called *nonhomogeneous*. Here, an attempt is made to extend the concepts used for the forced response of a single-degree-of-freedom system to the forced response of a general lumped-parameter system. In addition, the concept of stability of the forced response, as well as bounds on the forced response, is discussed. The beginning sections of this chapter are devoted to the solution for the forced response of a system by modal analysis, and the latter sections are devoted to introducing the use of a forced modal response in measurement and testing. The topic of experimental modal testing is considered in detail in Chapter 8. This chapter ends with an introduction to numerical simulation of the response to initial conditions and an applied force.

Since only linear systems are considered, the superposition principle can be employed. This principle states that the total response of the system is the sum of the free response (the homogeneous solution) plus the forced response (the nonhomogeneous solution). Hence, the form of the transient responses calculated in Chapter 3 are used again as part of the solution of a system subject to external forces and nonzero initial conditions. The numerical integration technique presented at the end of the chapter may also be used to simulate nonlinear system response, although that is not presented.

5.2 RESPONSE VIA STATE-SPACE METHODS

This section considers the state-space representation of a structure given by

$$\dot{\mathbf{x}}(t) = A\mathbf{x}(t) + \mathbf{f}(t) \tag{5.1}$$

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where A is a $2n \times 2n$ matrix containing the generalized mass, damping, and stiffness matrices as defined in Equation (2.20). The $2n \times 1$ state vector $\mathbf{x}(t)$ contains both the velocity and position vectors and will be referred to as the response in this case. Equation (5.1) reflects the fact that any set of n differential equations of second order can be written as a set of $2n$ first-order differential equations. In some sense, Equation (5.1) represents the most convenient form for solving for the forced response, since a great deal of attention has been focused on solving state-space descriptions numerically (such as the Runge–Kutta method), as discussed in Sections 1.10, 3.9, and 5.8, as well as analytically. In fact, several software packages are available for solving Equation (5.1) numerically on virtually every computing platform. The state-space form is also the form of choice for solving control problems (Chapter 7).

Only a few of the many approaches to solving this system are presented here; the reader is referred to texts on numerical integration and systems theory for other methods. More attention is paid in this chapter to developing methods that cater to the special form of mechanical systems, i.e., systems written in terms of position, velocity, and acceleration rather than in state space.

The first method presented here is simply that of solving Equation (5.1) by using the Laplace transform. Let $\mathbf{X}(0)$ denote the Laplace transform of the initial conditions. Taking the Laplace transform of Equation (5.1) yields

$$s\mathbf{X}(s) = A\mathbf{X}(s) + \mathbf{F}(s) + \mathbf{X}(0) \tag{5.2}$$

where $\mathbf{X}(s)$ denotes the Laplace transform of $\mathbf{x}(t)$ and is defined by

$$\mathbf{X}(s) = \int_0^\infty \mathbf{x}(t)\, e^{-st}\, dt \tag{5.3}$$

Here, s is a complex scalar. Algebraically solving Equation (5.2) for $\mathbf{X}(s)$ yields

$$\mathbf{X}(s) = (sI - A)^{-1}\mathbf{X}(0) + (sI - A)^{-1}\mathbf{F}(s) \tag{5.4}$$

The matrix $(sI - A)^{-1}$ is referred to as the *resolvent matrix*. The inverse Laplace transform of Equation (5.4) then yields the solution $\mathbf{x}(t)$. The form of Equation (5.4) clearly indicates the superposition of the transient solution, which is the first term on the right-hand side of Equation (5.4), and the forced response, which is the second term on the right-hand side of Equation (5.4). The inverse Laplace transform is defined by

$$\mathbf{x}(t) = \pounds^{-1}[X(s)] = \lim_{a\to\infty} \frac{1}{2\pi j} \int_{c-ja}^{c+ja} X(s)\, e^{st}\, ds \tag{5.5}$$

where $j = \sqrt{-1}$. In many cases, Equation (5.5) can be evaluated by using a table such as the one found in Thomson (1960) or a symbolic code. If the integral in Equation (5.5) cannot be found in a table or calculated, then numerical integration can be used to solve Equation (5.1) as presented in Section 5.8.

Example 5.2.1

Consider a simple single-degree-of-freedom system $\ddot{x} + 3\dot{x} + 2x = \mu(t)$, written in the state-space form defined by Equation (5.1) with

$$A = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}, \qquad \mathbf{f}(t) = \begin{bmatrix} f_1(t) \\ f_2(t) \end{bmatrix}, \qquad \mathbf{x}(t) = \begin{bmatrix} x(t) \\ \dot{x}(t) \end{bmatrix}$$

subject to a force given by $f_1 = 0$ and

$$f_2(t) = \begin{cases} 0, & t < 0 \\ 1, & t > 0 \end{cases}$$

the unit step function, and initial condition given by $\mathbf{x}(0) = [0 \quad 1]^T$. To solve this, first calculate

$$(sI - A) = \begin{bmatrix} s & -1 \\ 2 & s+3 \end{bmatrix}$$

and then determine the resolvent matrix

$$(sI - A)^{-1} = \frac{1}{s^2 + 3s + 2} \begin{bmatrix} s+3 & 1 \\ -2 & s \end{bmatrix}$$

Equation (5.4) becomes

$$\mathbf{X}(s) = \frac{1}{s^2 + 3s + 2} \begin{bmatrix} s+3 & 1 \\ -2 & s \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} + \frac{1}{s^2 + 3s + 2} \begin{bmatrix} s+3 & 1 \\ -2 & s \end{bmatrix} \begin{bmatrix} 0 \\ \frac{1}{s} \end{bmatrix} = \begin{bmatrix} \dfrac{s+1}{s^3 + 3s^2 + 2s} \\ \dfrac{s+1}{s^2 + 3s + 2} \end{bmatrix}$$

Taking the inverse Laplace transform by using a table yields

$$\mathbf{x}(t) = \begin{bmatrix} \frac{1}{2} - \frac{1}{2} e^{-2t} \\ e^{-2t} \end{bmatrix}$$

This solution is in agreement with the fact that the system is overdamped. Also note that $\dot{x}_1 = x_2$, as it should in this case, and that setting $t = 0$ satisfies the initial conditions.

A second method for solving Equation (5.1) imitates the solution of a first-order scalar equation, following what is referred to as the method of 'variation of parameters' (Boyce and DiPrima, 2005). For the matrix case, the solution depends on defining the exponential of a matrix. The *matrix exponential* of matrix A is defined by the infinite series

$$e^A = \sum_{k=0}^{\infty} \frac{A^k}{k!} \tag{5.6}$$

where $k!$ denotes the k factorial with $0! = 1$ and $A^0 = I$, the $n \times n$ identity matrix. This series converges for all square matrices A. By using the definition of Section 2.1 of a scalar multiplied by a matrix, the time-dependent matrix e^{At} is similarly defined as

$$e^{At} = \sum_{k=0}^{\infty} \frac{A^k t^k}{k!} \tag{5.7}$$

which also converges. The time derivative of Equation (5.7) yields

$$\frac{d}{dt}(e^{At}) = Ae^{At} = e^{At}A \tag{5.8}$$

Note that matrix A and the matrix e^{At} commute because a matrix commutes with its powers.

Following the method of variation of parameters, assume that the solution of Equation (5.1) is of the form

$$\mathbf{x}(t) = e^{At}\mathbf{c}(t) \tag{5.9}$$

where $\mathbf{c}(t)$ is an unknown vector function of time. The time derivative of Equation (5.9) yields

$$\dot{\mathbf{x}}(t) = Ae^{At}\mathbf{c}(t) + e^{At}\dot{\mathbf{c}}(t) \tag{5.10}$$

This results from the product rule and Equation (5.8). Substitution of Equation (5.9) into Equation (5.1) yields

$$\dot{\mathbf{x}}(t) = Ae^{At}\mathbf{c}(t) + \mathbf{f}(t) \tag{5.11}$$

Subtracting Equation (5.11) from Equation (5.10) yields

$$e^{At}\dot{\mathbf{c}}(t) = \mathbf{f}(t) \tag{5.12}$$

Premultiplying Equation (5.12) by e^{-At} (which always exists) yields

$$\dot{\mathbf{c}}(t) = e^{-At}\mathbf{f}(t) \tag{5.13}$$

Simple integration of this differential equation yields the solution for $\mathbf{c}(t)$:

$$\mathbf{c}(t) = \int_0^t e^{-A\tau}\mathbf{f}(\tau)d\tau + \mathbf{c}(0) \tag{5.14}$$

Here, the integration of a vector is defined as integration of each element of the vector, just as differentiation is defined on a per element basis. Substitution of Equation (5.14) into the assumed solution (5.9) produces the solution of Equation (5.1) as

$$\mathbf{x}(t) = e^{At}\int_0^t e^{-A\tau}\mathbf{f}(\tau)\mathrm{d}\tau + e^{At}\mathbf{c}(0) \tag{5.15}$$

Here, $\mathbf{c}(0)$ is the initial condition on $\mathbf{x}(t)$. That is, substitution of $t = 0$ into (5.9) yields

$$\mathbf{x}(0) = e^{0}\mathbf{c}(0) = I\mathbf{c}(0) = \mathbf{c}(0)$$

so that $\mathbf{c}(0) = \mathbf{x}(0)$, the initial conditions on the state vector. The complete solution of Equation (5.1) can then be written as

$$\mathbf{x}(t) = e^{At}\mathbf{x}(0) + \int_0^t e^{A(t-\tau)}\mathbf{f}(\tau)d\tau \tag{5.16}$$

The first term represents the response due to the initial conditions, i.e., the free response of the system. The second term represents the response due to the applied force, i.e., the steady state response. Note that the solution given in Equation (5.16) is independent of the nature of the viscous damping in the system (i.e., proportional or not) and gives both the displacement and velocity time response.

The matrix e^{At} is often called the *state transition matrix* of the system defined by Equation (5.1). Matrix e^{At} 'maps' the initial condition $\mathbf{x}(0)$ into the new or next position $\mathbf{x}(t)$. While Equation (5.16) represents a closed-form solution of Equation (5.1) for any state matrix A, use of this form centers on the calculation of matrix e^{At}. Many papers have been written on different methods of calculating e^{At} (Moler and Van Loan, 1978).

One method of calculating e^{At} is to realize that e^{At} is equal to the inverse Laplace transform of the resolvent matrix for A. In fact, a comparison of Equations (5.16) and (5.4) yields

$$e^{At} = £^{-1}\{(sI - A)^{-1}\} \tag{5.17}$$

Another interesting method of calculating e^{At} is restricted to those matrices A with diagonal Jordan form. Then it can be shown [recall Equation (3.31)] that

$$e^{At} = Ue^{\Lambda t}U^{-1} \tag{5.18}$$

where U is the matrix of eigenvectors of A, and Λ is the diagonal matrix of eigenvalues of A. Here, $e^{\Lambda t} = \text{diag}\left[e^{\lambda_1 t}\ e^{\lambda_2 t} \cdots e^{\lambda_n t}\right]$, where the λ_i denote the eigenvalues of matrix A.

Example 5.2.2

Compute the matrix exponential e^{At} for the state matrix

$$A = \begin{bmatrix} -2 & 3 \\ -3 & -2 \end{bmatrix}$$

Using the Laplace transform approach of Equation (5.18) requires forming of the matrix $(sI - A)$:

$$sI - A = \begin{bmatrix} s+2 & -3 \\ 3 & s+2 \end{bmatrix}$$

Calculating the inverse of this matrix yields

$$(sI - A)^{-1} = \begin{bmatrix} \dfrac{s+2}{(s+2)^2+9} & \dfrac{3}{(s+2)^2+9} \\ \dfrac{-3}{(s+2)^2+9} & \dfrac{s+2}{(s+2)^2+9} \end{bmatrix}$$

The matrix exponential is now computed by taking the inverse Laplace transform of each element. This results in

$$e^{At} = £^{-1}\{(sI - A)^{-1}\} = \begin{bmatrix} e^{-2t}\cos 3t & e^{-2t}\sin 3t \\ -e^{-2t}\sin 3t & e^{-2t}\cos 3t \end{bmatrix} = e^{-2t}\begin{bmatrix} \cos 3t & \sin 3t \\ -\sin 3t & \cos 3t \end{bmatrix}$$

5.3 DECOUPLING CONDITIONS AND MODAL ANALYSIS

An alternative approach to solving for the response of a system by transform or matrix exponent methods is to use the eigenvalue and eigenvector information from the free response as a tool for solving for the forced response. This provides a useful theoretical tool as well as a computationally different approach. Approaches based on the eigenvectors of the system are referred to as *modal analysis* and also form the basis for understanding modal test methods (see Chapter 8). Modal analysis can be carried out in either the state vector coordinates of first-order form or the physical coordinates defining the second-order form.

First consider the system described by Equation (5.1). If matrix A has a diagonal Jordan form (Section 3.4), which happens when it has distinct eigenvalues, for example, then matrix A can be diagonalized by its modal matrix. In this circumstance, Equation (5.1) can be reduced to $2n$ independent first-order equations. To see this, let $\mathbf{u}_i$ be the eigenvectors of the state matrix A with eigenvalues λ_i. Let $U = [\mathbf{u}_1 \ \mathbf{u}_2 \cdots \mathbf{u}_{2n}]$ be the modal matrix of matrix A. Then substitute $\mathbf{x} = U\mathbf{z}$ into Equation (5.1) to obtain

$$U\dot{\mathbf{z}} = AU\mathbf{z} + \mathbf{f} \tag{5.19}$$

Premultiplying Equation (5.19) by U^{-1} yields the decoupled system

$$\dot{\mathbf{z}} = U^{-1}AU\mathbf{z} + U^{-1}\mathbf{f} \tag{5.20}$$

each element of which is of the form

$$\dot{z}_i = \lambda_i z_i + F_i \tag{5.21}$$

where z_i is the ith element of the vector $\mathbf{z}$ and F_i is the ith element of the vector $\mathbf{F} = U^{-1}\mathbf{f}$. Equation (5.21) can now be solved using scalar integration of each of the equations subject to the initial condition $z_i(0) = [U^{-1}\mathbf{x}(0)]_i$. In this way, the vector $\mathbf{z}(t)$ can be calculated, and the solution $\mathbf{x}(t)$ in the original coordinates becomes

$$\mathbf{x}(t) = U\mathbf{z}(t) \tag{5.22}$$

The amount of effort required to calculate the solution via this method is comparable with that required to calculate the solution via Equation (5.17). The modal form offered by Equations (5.21) and (5.22) provides a tremendous analytical advantage.

The above process can also be used on systems described in the second-order form. The differences are that the eigenvector–eigenvalue problem is solved in n dimensions instead of $2n$ dimensions and the solution is the position vector instead of the state vector of position and velocity. In addition, the modal vectors used to decouple the equations of motion have important physical significance when viewed in second-order form which is not as apparent in the state-space form.

Consider examining the forced response in the physical or spatial coordinates defined by Equation (2.7). First consider the simplest problem, that of calculating the forced response of an undamped nongyroscopic system of the form

$$M\ddot{\mathbf{q}}(t) + K\mathbf{q}(t) = \mathbf{f}(t) \tag{5.23}$$

where M and K are $n \times n$ real positive definite matrices, $\mathbf{q}(t)$ is the vector of displacements, and $\mathbf{f}(t)$ is a vector of applied forces. The system is also subject to an initial position given by $\mathbf{q}(0)$ and an initial velocity given by $\dot{\mathbf{q}}(0)$.

To solve Equation (5.23) by eigenvector expansions, one must first solve the eigenvalue–eigenvector problem for the corresponding homogeneous system. That is, one must calculate λ_i and $\mathbf{u}_i$ such that $(\lambda_i = \omega_i^2)$

$$\lambda_i \mathbf{u}_i = \tilde{K}\mathbf{u}_i \tag{5.24}$$

Note that $\mathbf{u}_i$ now denotes an $n \times 1$ eigenvector of the mass normalized stiffness matrix. From this, the modal matrix S_m is calculated and normalized such that

$$\begin{aligned} S_m^T M S_m &= I \\ S_m^T K S_m &= \Lambda = \operatorname{diag}(\omega_i^2) \end{aligned} \tag{5.25}$$

This procedure is the same as that of Section 3.3 except that, in the case of the forced response, the form that the temporal part of the solution will take is not known. Hence, rather than assuming that the dependence is of the form $\sin(\omega t)$, the temporal form is computed from a generic temporal function designated as $y_i(t)$.

Since the eigenvectors $\mathbf{u}_i$ form a basis in an n-dimensional space, any vector $\mathbf{q}(t_1)$, where t_1 is some fixed but arbitrary time, can be written as a linear combination of the vectors $\mathbf{u}_i$; thus

$$\mathbf{q}(t_1) = \sum_{i=1}^{n} y_i(t_1)\mathbf{u}_i = S_m \mathbf{y}(t_1) \tag{5.26}$$

where $\mathbf{y}(t_1)$ is an n-vector with components $y_i(t_1)$ to be determined. Since t_1 is arbitrary, it is reasoned that Equation (5.25) must hold for any t. Therefore

$$\mathbf{q}(t) = \sum_{i=1}^{n} y_i(t)\mathbf{u}_i = S_m \mathbf{y}(t), \qquad t \geq 0 \tag{5.27}$$

This must be true for any n-dimensional vector $\mathbf{q}$. In particular this must hold for solutions of Equation (5.23). Substitution of Equation (5.27) into Equation (5.23) shows that the vector $\mathbf{y}(t)$ must satisfy

$$MS_m \ddot{\mathbf{y}}(t) + KS_m \mathbf{y}(t) = \mathbf{f}(t) \tag{5.28}$$

Premultiplying by S_m^T yields

$$\ddot{\mathbf{y}}(t) + \Lambda \mathbf{y}(t) = S_m^T \mathbf{f}(t) \tag{5.29}$$

Equation (5.28) represents n decoupled equations, each of the form

$$\ddot{\mathbf{y}}_i(t) + \omega_i^2 \mathbf{y}_i(t) = f_i(t) \tag{5.30}$$

where $f_i(t)$ denotes the ith element of the vector $S_m^T \mathbf{f}(t)$.

If K is assumed to be positive definite, each ω_i^2 is a positive real number. Denoting the 'modal' initial conditions by $y_i(0)$ and $\dot{y}_i(0)$, the solutions of Equation (5.30) are calculated by the method of variation of parameters to be

$$y_i(t) = \frac{1}{\omega_i}\int_0^t f_i(t-\tau)\sin(\omega_i\tau)\,d\tau + y_i(0)\cos(\omega_i t) + \frac{\dot{y}_i(0)}{\omega_i}\sin(\omega_i t), \qquad i = 1, 2, 3, \ldots, n \tag{5.31}$$

(see Boyce and DiPrima, 2005, for a derivation).

If K is semidefinite, one or more values of ω_i^2 might be zero. Then, Equation (5.30) would become

$$\ddot{y}_i(t) = f_i(t) \tag{5.32}$$

Integrating Equation (5.32) then yields

$$y_i(t) = \int_0^t \left[\int_0^\tau f_i(s)\,ds\right] d\tau + y_i(0) + \dot{y}_i(0)t \tag{5.33}$$

which represents a rigid body motion.

The initial conditions for the new coordinates are determined from the initial conditions for the original coordinates by the transformation

$$\mathbf{y}(0) = S_m^{-1}\mathbf{q}(0) \tag{5.34}$$

and

$$\dot{\mathbf{y}}(0) = S_m^{-1}\dot{\mathbf{q}}(0) \tag{5.35}$$

This method is often referred to as *modal analysis* and differs from the state-space modal approach in that the computations involve matrices and vectors of size n rather than $2n$. They result in a solution for the position vector rather than the $2n$-dimensional state vector. The coordinates defined by the vector $\mathbf{y}$ are called *modal coordinates, normal coordinates, decoupled coordinates*, and (sometimes) *natural coordinates*. Note that, in the case of a free response, i.e., $f_i = 0$, then $y_i(t)$ is just $e^{\pm\omega_i jt}$, where ω_i is the ith natural frequency of the system, as discussed in Section 3.3.

Alternatively, the modal decoupling described in the above paragraphs can be obtained by using the mass normalized stiffness matrix. To see this, substitute $\mathbf{q} = M^{1/2}\mathbf{r}$ into Equation (2.11), multiply by $M^{-1/2}$ to form $\tilde{K} = M^{-1/2}KM^{-1/2}$, compute the normalized eigenvectors of $\tilde{K}$, and use these to form the columns of the orthogonal matrix S. Next, use the substitution $\mathbf{r} = S\mathbf{y}$ in the equation of motion, premultiply by S^T, and equation (5.30) results. This procedure is illustrated in the following example.

Example 5.3.1

Consider the undamped system of example 3.3.2 and Figure 2.4 subject to a harmonic force applied to m_2, given by

$$\mathbf{f}(t) = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \sin 3t$$

Following the alternative approach, compute the modal force by multiplying the physical force by $S^T M^{-1/2}$:

$$S^T M^{-1/2}\mathbf{f}(t) = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 1/3 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix}\sin 3t = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}\sin 3t$$

Combining this with the results of example 3.32 leads to the modal equations [Equation (5.30)]

$$\ddot{y}_1(t) + 2y_1(t) = \frac{1}{\sqrt{2}}\sin 3t$$

$$\ddot{y}_2(t) + 4y_2(t) = \frac{1}{\sqrt{2}}\sin 3t$$

This of course is subject to the transformed initial conditions, and each equation can be solved by the methods of Chapter 1. For instance, if the initial conditions in the physical coordinates are

$$\mathbf{q}(0) = \begin{bmatrix} 0.1 \\ 0 \end{bmatrix} \quad \text{and} \quad \dot{\mathbf{q}}(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

then in the modal coordinates the initial velocity remains zero but the initial displacement is transformed (solving $\mathbf{q} = M^{-1/2}\,\mathbf{r}$ and $\boldsymbol{r} = S\mathbf{y}$, for $\mathbf{y}$) to become

$$\mathbf{y}(0) = S^T M^{1/2}\mathbf{q}(0) = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 0.1 \\ 0 \end{bmatrix} = \frac{0.3}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

Solving for y_1 proceeds by first computing the particular solution (see Section 1.4 with zero damping) by assuming that $y(t) = X\sin 3t$ in the modal equation for y_1 to obtain

$$-9X\sin 3t + 2X\sin 3t = \frac{1}{\sqrt{2}}\sin 3t$$

Solving for $\mathbf{X}$ leads to the particular solution $y_{1\mathrm{p}}(t) = \left(-1/(7\sqrt{2})\right)\sin 3t$. The total solution (from Equation (1.21) with zero damping) is then

$$y_1(t) = a\sin\sqrt{2}t + b\cos\sqrt{2}t - \frac{1}{7\sqrt{2}}\sin 3t$$

Applying the modal initial conditions yields

$$y_1(0) = b = \frac{0.3}{\sqrt{2}}, \qquad \dot{y}_1(0) = \sqrt{2}a - \frac{3}{7\sqrt{2}} = 0 \Rightarrow a = \frac{3}{14}$$

Thus, the solution of the first modal equation is

$$y_1(t) = \frac{3}{14}\sin\sqrt{2}t + \frac{0.3}{\sqrt{2}}\cos\sqrt{2}t - \frac{1}{7\sqrt{2}}\sin 3t$$

Likewise, the solution to the second modal equation is

$$y_2(t) = \frac{0.3}{\sqrt{2}}\sin 2t - \frac{0.3}{\sqrt{2}}\cos 2t - \frac{0.2}{\sqrt{2}}\sin 3t$$

The solution in physical coordinates is then found from $\mathbf{q}(t) = M^{-1/2}S\mathbf{y}(t)$.

5.4 RESPONSE OF SYSTEMS WITH DAMPING

The key to using modal analysis to solve for the forced response of systems with velocity-dependent terms is whether or not the system can be decoupled. As in the case of the free response, discussed in Section 3.5, this will happen for symmetric systems if and only if the coefficient matrices commute, i.e., if $KM^{-1}D = DM^{-1}K$. Ahmadian and Inman (1984a) reviewed previous work on decoupling and extended the commutivity condition to systems with asymmetric coefficients. Inman (1982) and Ahmadian and Inman (1984b) used the decoupling condition to carry out modal analysis for general asymmetric systems with commuting coefficients. In each of these cases the process is the same, with an additional transformation into symmetric coordinates, as introduced in Section 4.9. Hence, only the symmetric case is illustrated here.

To this end, consider the problem of calculating the forced response of the nongyroscopic damped linear system given by

$$M\ddot{\mathbf{q}} + D\ddot{\mathbf{q}} + K\mathbf{q} = \mathbf{f}(t) \tag{5.36}$$

where M and K are symmetric and positive definite and D is symmetric and positive semidefinite. In addition, it is assumed that $KM^{-1}D = DM^{-1}K$. Let S_m be the modal matrix of K normalized with respect to M, as defined by Equations (3.69) and (3.70). Then, the commutivity of the coefficient matrices yields

$$\begin{aligned} S_m^T M S_m &= I \\ S_m^T D S_m &= \Lambda_D = \mathrm{diag}(2\zeta_i\omega_i) \\ S_m^T K S_m &= \Lambda_K = \mathrm{diag}(\omega_i^2) \end{aligned} \tag{5.37}$$

where Λ_D and Λ_K are diagonal matrices, as indicated. Making the substitution $\mathbf{q} = \mathbf{q}(t) = S_m\mathbf{y}(t)$ in Equation (5.36) and premultiplying by S_m^T as before yields

$$I\ddot{\mathbf{y}} + \Lambda_D\dot{\mathbf{y}} + \Lambda_K\mathbf{y} = S_m^T\mathbf{f}(t) \tag{5.38}$$

Equation (5.38) is diagonal and can be written as n decoupled equations of the form

$$\ddot{\mathbf{y}}_i(t) + 2\zeta_i\omega_i\dot{y}_i(t) + \omega_i^2 y_i(t) = f_i(t) \tag{5.39}$$

Here, $\zeta_i = \lambda_i(D)/2\omega_{\mathrm{i}}$, where $\lambda_i(D)$ denotes the eigenvalues of matrix D. In this case these are the nonzero elements of Λ_D. This expression is the nonhomogeneous counterpart of Equation (3.71).

If it is assumed that $4\tilde{K} - \tilde{D}^2$ is positive definite, then $0 < \zeta_i < 1$, and the solution of Equation (5.39) (assuming all initial conditions are zero) is

$$y_i(t) = \frac{1}{\omega_{di}} \int_0^t e^{-\zeta_i\omega_i\tau} f_i(t-\tau)\sin(\omega_{di}\tau)\,d\tau, \qquad i = 1, 2, 3, \ldots, n \tag{5.40}$$

where $\omega_{di} = \omega_i\sqrt{1 - \zeta_i^{(2)}}$. If $4\tilde{K} - \tilde{D}^2$ is not positive definite, other forms of $y_i(t)$ result, depending on the eigenvalues of the matrix $4\tilde{K} - \tilde{D}^2$, as discussed in Section 3.6.

In addition to the forced response given by Equation (5.40), there will be a transient response, or homogeneous solution, due to any nonzero initial conditions. If this response is denoted by y_i^H, then the total response of the system in the decoupled coordinate system is the sum $y_i(t) + y_i^H(t)$. This solution is related to the solution in the original coordinates by the modal matrix S_m and is given by

$$\mathbf{q}(t) = S_m[\mathbf{y}(t) + \mathbf{y}^H(t)] \tag{5.41}$$

For asymmetric systems, the procedure is similar, with the exception of computing a second transformation; this transforms the asymmetric system into an equivalent symmetric system, as done in Section 4.9.

For systems in which the coefficient matrices do not commute, i.e., for which $KM^{-1}D \neq DM^{-1}K$ in Equation (5.36), modal analysis of a sort is still possible without resorting to state space. To this end, consider the symmetric case given by the system

$$M\ddot{\mathbf{q}} + D\dot{\mathbf{q}} + K\mathbf{q} = \mathbf{f}(t) \tag{5.42}$$

where M, D, and K are symmetric.

Let $\mathbf{u}_i$ be the eigenvectors of the lambda matrix

$$(M\lambda_i^2 + D\lambda_i + K)\mathbf{u}_i = \mathbf{0} \tag{5.43}$$

with associated eigenvalues λ_i. Let n be the number of degrees of freedom (so there are $2n$ eigenvalues), let $2s$ be the number of real eigenvalues, and let $2(n - s)$ be the number of complex eigenvalues. Assuming that $D_2(\lambda)$ is simple and that the $\mathbf{u}_i$ are normalized so that

$$\mathbf{u}_i^T(2M\lambda_i + D)\mathbf{u}_i = 1 \tag{5.44}$$

a particular solution of Equation (5.42) is given by Lancaster (1966) in terms of the generalized modes $\mathbf{u}_i$ to be

$$\mathbf{q}(t) = \sum_{k=1}^{2s} \mathbf{u}_k\mathbf{u}_k^T \int_0^t e^{-\lambda_k(t+\tau)}\mathbf{f}(\tau)\,d\tau + \sum_{k=2s+1}^{2n} \int_0^t \mathrm{Re}\{e^{\lambda_k(t-\tau)}\mathbf{u}_k\mathbf{u}_k^T\}\mathbf{f}(\tau)\,d\tau \tag{5.45}$$

This expression is more difficult to compute but does offer some insight into the form of the solution that is useful in modal testing, as will be illustrated in Chapter 8. Note that the eigenvalues indexed λ_1 through λ_{2s} are real, whereas those labeled λ_{2s+1} through λ_{2n} are complex. The complex eigenvalues λ_{2s+1} and $\lambda_{2(s+1)}$ are conjugates of each other. Also, note that the nature of the matrices D and K completely determines the value of s. In fact, if $4\tilde{K} - \tilde{D}^2$ is positive definite, $s = 0$ in Equation (5.44). Also note that, if λ_k is real, so is the corresponding $\mathbf{u}_k$. On the other hand, if λ_k is complex, the corresponding eigenvector $\mathbf{u}_k$ is real if and only if $KM^{-1}D = DM^{-1}K$; otherwise, the eigenvectors are complex valued.

The particular solution (5.45) has the advantage of being stated in the original, or physical, coordinate system. To obtain the total solution, the transient response developed in Section (3.4) must be added to Equation (5.45). This should be done unless steady state conditions prevail.

Example 5.4.1

Consider example 5.3.1 with a damping force applied of the form $D = 0.1K$ (proportional damping). In this case $4\tilde{K} - \tilde{D}$ is positive definite so that each mode will be underdamped. The alternative transformation used in example 5.3.1 is employed here to find the modal equations given in Equation (5.39). The equations of motion and initial conditions (assuming compatible units) are

$$\begin{bmatrix} 9 & 0 \\ 0 & 1 \end{bmatrix}\ddot{\mathbf{q}}(t) + \begin{bmatrix} 2.7 & -0.3 \\ -0.3 & 0.3 \end{bmatrix}\dot{\mathbf{q}}(t) + \begin{bmatrix} 27 & -3 \\ -3 & 3 \end{bmatrix}\mathbf{q}(t) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}\sin 3t, \quad \mathbf{q}(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \dot{\mathbf{q}}(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Since the damping is proportional, the undamped transformation computed in examples 3.3.2 and 5.3.1 can be used to decouple the equations of motion. Using the transformation $\mathbf{y}(t) = S^T M^{1/2}\mathbf{q}(t)$ and premultiplying the equations of motion by $S^T M^{-1/2}$ yields the uncoupled modal equations

$$\ddot{y}_1(t) + 0.2\dot{y}_1(t) + 2y_1(t) = \frac{1}{\sqrt{2}}\sin 3t, \qquad y_1(0) = \frac{3}{\sqrt{2}}, \qquad \dot{y}_1(0) = 0$$

$$\ddot{y}_2(t) + 0.4\dot{y}_2(t) + 4y_2(t) = \frac{1}{\sqrt{2}}\sin 3t, \qquad y_2(0) = \frac{-3}{\sqrt{2}}, \qquad \dot{y}_2(0) = 0$$

From the modal equations the frequencies and damping ratios are evident:

$$\omega_1 = \sqrt{2} = 1.414\,\text{rad/s}, \qquad \zeta_1 = \frac{0.2}{2\sqrt{2}} = 0.071 < 1, \qquad \omega_{d_1} = \omega_1\sqrt{1-\zeta_1^2} = 1.41\,\text{rad/s}$$

$$\omega_2 = \sqrt{4} = 2\,\text{rad/s}, \qquad \zeta_2 = \frac{0.4}{(2)(2)} = 0.1 < 1, \qquad \omega_{d_2} = \omega_2\sqrt{1-\zeta_2^2} = 1.99\,\text{rad/s}$$

Solving the two modal equations using the approach of example 1.4.1 (y_2 is solved there) yields

$$y_1(t) = e^{-0.1t}(0.3651\sin 1.41t + 2.1299\cos 1.41t) - 0.1015\sin 3t - 0.0087\cos 3t$$
$$y_2(t) = -e^{-0.2t}(0.0084\sin 1.99t + 2.0892\cos 1.99t) - 0.1325\sin 3t - 0.032\cos 3t$$

This forms the solution in modal coordinates. To regain the solution in physical coordinates, use the transformation $\mathbf{q}(t) = M^{-1/2}S\mathbf{y}(t)$. Note that the transient term is multiplied by a decaying exponential in time and will decay off, leaving the steady state to persist.

5.5 BOUNDED-INPUT, BOUNDED-OUTPUT STABILITY

In the previous chapter, several types of stability for the free response of a system were defined and discussed in great detail. In this section the concept of stability as it applies to the forced response of a system is discussed. In particular, systems are examined in the state-space form given by Equation (5.1).

The stability of the forced response of a system is defined in terms of bounds of the response vector $\mathbf{x}(t)$. Hence, it is important to recall that a vector $\mathbf{x}(t)$ is bounded if

$$\|\mathbf{x}(t)\| = \sqrt{\mathbf{x}^T\mathbf{x}} < M \tag{5.46}$$

where M is some finite positive real number. The quantity $||\mathbf{x}||$ just defined is called the *norm* of $\mathbf{x}(t)$. The response $\mathbf{x}(t)$ is also referred to as the *output* of the system, hence the phrase bounded-output stability.

A fundamental classification of stability of forced systems is called *bounded-input, bounded-output* (BIBO) *stability*. The system described by Equation (5.1) is called BIBO stable if any bounded forcing function $\mathbf{f}(t)$, called the input, produces a bounded response $\mathbf{x}(t)$, i.e., a bounded output, regardless of the bounded initial condition $\mathbf{x}(0)$. An example of a system that is not BIBO stable is given by the single-degree-of-freedom oscillator

$$\ddot{y} + \omega_n^2 y = \sin \omega t \tag{5.47}$$

In state-space form this becomes

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 \\ -\omega_n^2 & 0 \end{bmatrix} \mathbf{x} + \mathbf{f}(t) \tag{5.48}$$

where $\mathbf{x} = [y \quad \dot{y}]^T$ and $\mathbf{f}(t) = [0 \quad \sin \omega t]^T$. This system is not BIBO stable, since, for a bounded input $y(t)$, and hence $\mathbf{x}(t)$, blows up when $\omega = \omega_n$ (i.e., at resonance).

A second classification of stability is called *bounded stability*, or *Lagrange stability*, and is a little weaker than BIBO stability. The system described in Equation (5.1) is said to be *Lagrange stable* with respect to a *given* input $\mathbf{f}(t)$ if the response $\mathbf{x}(t)$ is bounded for any bounded initial condition $\mathbf{x}(0)$. Referring to the example of the previous paragraph, if $\omega \neq \omega_n$, then the system described by Equation (5.48) is bounded with respect to $\mathbf{f}(t) = [0 \quad \sin \omega t]^T$ because, when $\omega \neq \omega_n$, $\mathbf{x}(t)$ does not blow up. Note that, if a given system is BIBO stable, it will also be Lagrange stable. However, a system that is Lagrange stable may not be BIBO stable.

As an example of a system that is BIBO stable, consider adding damping to the preceding system. The result is a single-degree-of-freedom damped oscillator that has the state matrix

$$A = \begin{bmatrix} 0 & 1 \\ -k/m & -c/m \end{bmatrix} \tag{5.49}$$

Recall that the damping term prevents the solution $\mathbf{x}(t)$ from becoming unbounded at resonance. Hence, $y(t)$ and $\dot{y}(t)$ are bounded for any bounded input $\mathbf{f}(t)$, and the system is BIBO stable as well as Lagrange stable.

The difference in the two examples is due to the stability of the free response of each system. The undamped oscillator is stable but not asymptotically stable, and the forced response *is not* BIBO stable. On the other hand, the damped oscillator is asymptotically stable and *is* BIBO stable. To some extent this is true in general. Namely, it is shown by Müller and Schiehlen (1977) that, if the forcing function $\mathbf{f}(t)$ can be written as a constant matrix B times a vector $\mathbf{u}$, i.e., $\mathbf{f}(t) = B\mathbf{u}$, then, if

$$\text{rank}\begin{bmatrix} B & AB & A^2B & A^3B \cdots A^{2n-1}B \end{bmatrix} = 2n \tag{5.50}$$

where $2n$ is the dimension of matrix A, the system in Equation (5.1) is BIBO stable if and only if the free response is asymptotically stable. If $\mathbf{f}(t)$ does not have this form or does not satisfy the rank condition (5.50), then asymptotically stable systems are BIBO stable, and BIBO stable systems have a stable free response.

Another way to look at the difference between the above two examples is to consider the phenomenon of resonance. The undamped single-degree-of-freedom oscillator of Equation (5.48) experiences an infinite amplitude at $\omega = \omega_n$, the resonance condition, which is certainly unstable. However, the underdamped single-degree-of-freedom oscillator of Equation (5.49) is bounded at the resonance condition, as discussed in Section 1.4. Hence, the damping 'lowers' the peak response at resonance from infinity to some finite, or bounded, value, resulting in a system that is BIBO stable.

The obvious use of the preceding conditions is to use the stability results of Chapter 4 for the free response to guarantee BIBO stability or boundedness of the forced response, $\mathbf{x}(t)$. To this extent, other concepts of stability of systems subject to external forces are not developed. Instead, some specific bounds on the forced response of a system are examined in the next section.

5.6 RESPONSE BOUNDS

Given that a system is either BIBO stable or at least bounded, it is sometimes of interest to calculate bounds for the forced response of the system without actually calculating the response itself. A summary of early work on the calculation of bounds is given in review papers by Nicholson and Inman (1983) and Nicholson and Lin (1996). More recent work is given in Hu and Eberhard (1999). A majority of the work reported there examines bounds for systems in the physical coordinates $\mathbf{q}(t)$ in the form of Equation (5.36). In particular, if $DM^{-1}K = KM^{-1}D$ and if the forcing function or input is of the form (periodic)

$$\mathbf{f}(t) = \mathbf{f}_0 e^{j\omega t} \tag{5.51}$$

where $\mathbf{f}_0$ is an $n \times 1$ vector of constants, $j^2 = -1$, and ω is the driving frequency, then

$$\frac{||\mathbf{q}(t)||}{||\mathbf{f}_0||} \leqslant \max_j \begin{cases} \dfrac{1}{\lambda_j(K)} & \text{if } \dfrac{\lambda_j(K)}{\lambda_j(M)} < \dfrac{\lambda_j^2(D)}{2\lambda_j^2(M)} \\ \sqrt{\dfrac{\lambda_i(M)}{\lambda_i^2(D)\lambda_i(K)}} & \text{otherwise} \end{cases} \tag{5.52}$$

Here, $\lambda_i(M)$, $\lambda_i(D)$, and $\lambda_i(K)$ are used to denote the ordered eigenvalues of the matrices M, D, and K respectively. The first inequality in expression (5.52) is satisfied if the free system is overdamped, and the bound $[\lambda_i^2(D)\lambda_i(K)/\lambda_i(M)]^{-1/2}$ is applied for underdamped systems.

Bounds on the forced response are also available for systems that do not decouple, i.e., for systems with coefficient matrices such that $DM^{-1}K \neq KM^{-1}D$. One way to approach such systems is to write the system in the normal coordinates of the undamped system. Then the resulting damping matrix can be written as the sum of a diagonal matrix and an off-diagonal matrix, which clearly indicates the degree of decoupling.

Substituting $\mathbf{q}(t) = S_m\mathbf{x}$ into Equation (5.36) and premultiplying by S_m^T, where S_m is the modal matrix for K, yields

$$I\ddot{\mathbf{x}} + (\Lambda_D + D_1)\dot{\mathbf{x}} + \Lambda_K \mathbf{x} = \tilde{\mathbf{f}}(t) \tag{5.53}$$

The matrix Λ_D is the diagonal part of $S_m^T D S_m$, D_1 is the matrix of off-diagonal elements of $S_m^T D S_m$, Λ_K is the diagonal matrix of squared undamped natural frequencies, and $\tilde{\mathbf{f}} = S_m^T \mathbf{f}$.

The steady state response of Equation (5.53) with a sinusoidal input, i.e., $\tilde{\mathbf{f}} = \mathbf{f}_0 e^{j\omega t}$ and Equation (5.53) underdamped, is given by

$$\frac{||\mathbf{q}(t)||}{||\mathbf{f}_0||} < \frac{2}{\lambda_{\min}(\Lambda_D \Lambda_C)} e^{(\beta ||D_1||/\lambda_{\min})} \tag{5.54}$$

Here, $\lambda_{\min}(\Lambda_D \Lambda_C)$ denotes the smallest eigenvalue of the matrix $\Lambda_D \Lambda_C$, where $\Lambda_C = (4\Lambda_K - \Lambda_D^2)^{1/2}$. Also, $\lambda_{\min}$ is the smallest eigenvalue of the matrix Λ_D, $||D_1||$ is the matrix norm defined by the maximum value of the square root of the largest eigenvalue of $D_1^T D_1 = D_1^2$, and β is defined by

$$\beta = \sqrt{\left\| I + (\Lambda_C^{-1} \Lambda_D)^2 \right\|} \tag{5.55}$$

Examination of the bound in Equation (5.53) shows that, the greater the coupling in the system characterized by $||D_1||$, the larger is the bound. Thus, for small values of $||D_1||$, i.e., small coupling, the bound is good, whereas for large values of $||D_1||$ or very highly coupled systems, the bound will be very large and too conservative to be of practical use. This is illustrated in example 5.6.1.

Example 5.6.1

Consider a system defined by Equation (5.36) with $M = I$:

$$K = \begin{bmatrix} 5 & -1 \\ -1 & 1 \end{bmatrix}, \qquad D = 0.5K + 0.5I + \xi \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

subject to a sinusoidal driving force applied to the first mass so that $\mathbf{f}_0 = [1 \quad 0]^T$. The parameter ξ clearly determines the degree of proportionality or coupling in the system. The bounds are tabulated in Table 5.1 for various values of ξ, along with a comparison with the exact solution.

Examination of Table 5.1 clearly illustrates that, as the degree of coupling increases (larger ξ), the bound gets farther away from the actual response. Note that the value given in the 'exact' column is the largest value obtained by the exact response.

Table 5.1 Forced response bounds.

ξ	Exact solution	Bound
0	1.30	1.50
−0.1	1.56	2.09
−0.2	1.62	3.05
−0.3	1.70	4.69
−0.4	1.78	8.60

5.7 FREQUENCY RESPONSE METHODS

This section attempts to extend the concept of frequency response introduced in Sections 1.5 and 1.6 to multiple-degree-of-freedom systems. In so doing, the material in this section makes the connection between analytical modal analysis and experimental modal analysis discussed in Chapter 8. The development starts by considering the response of a structure to a harmonic or sinusoidal input, denoted by $\mathbf{f}(t) = \mathbf{f}_0 e^{j\omega t}$. The equations of motion in spatial or physical coordinates given by Equation (5.36) with no damping ($D = 0$) or Equation (5.23) are considered first. In this case, an oscillatory solution of Equation (5.23) of the form

$$\mathbf{q}(t) = \mathbf{u}e^{j\omega t} \tag{5.56}$$

is assumed. This is equivalent to the frequency response theorem stated in Section 1.5. That is, if a system is harmonically excited, the response will consist of a steady state term that oscillates at the driving frequency with different amplitude and phase.

Substitution of the assumed oscillatory solution into Equation (5.36) with $D = 0$ yields

$$\left(K - \omega^2 M\right)\mathbf{u}e^{j\omega t} = \mathbf{f}_0 e^{j\omega t} \tag{5.57}$$

Dividing through by the nonzero scalar $e^{j\omega t}$ and solving for $\mathbf{u}$ yields

$$\mathbf{u} = (K - \omega^2 M)^{-1}\mathbf{f}_0 \tag{5.58}$$

Note that the matrix inverse of $(K - \omega^2 M)$ exists as long as ω is not one of the natural frequencies of the structure. This is consistent with the fact that, without damping, the system is Lagrange stable and not BIBO stable. The matrix coefficient of Equation (5.58) is defined as the *receptance matrix*, denoted by $\alpha(\omega)$, i.e.,

$$\alpha(\omega) = (K - \omega^2 M)^{-1} \tag{5.59}$$

Equation (5.58) can be thought of as the *response model* of the structure. Solution of Equation (5.58) yields the vector $\mathbf{u}$, which, coupled with Equation (5.56), yields the steady state response of the system to the input force $\mathbf{f}(t)$.

Each element of the response matrix can be related to a single-frequency response function by examining the definition of matrix multiplication. In particular, if all the elements of the vector $\mathbf{f}_0$, denoted by f_i, except the jth element are set equal to zero, then the ijth element of $\alpha(\omega)$ is just the receptance transfer function between u_i, the ith element of the response vector $\mathbf{u}$, and f_j. That is

$$\alpha_{ij}(\omega) = \frac{u_i}{f_j}, \qquad f_i = 0, \qquad i = 0, \ldots, n, \qquad i \neq j \tag{5.60}$$

Note that, since $\alpha(\omega)$ is symmetric, this interpretation implies that $u_i/f_j = u_j/f_i$. Hence, a force applied at position j yields the same response at point i as a force applied at i does at point j. This is called *reciprocity*.

An alternative to computing the inverse of the matrix $(K - \omega^2 M)$ is to use the modal decomposition of $\alpha(\omega)$. Recalling Equations (3.20) and (3.21) from Section 3.2, the matrices M and K can be rewritten as

$$M = S_m^{-T} S_m^{-1} \tag{5.61}$$

$$K = S_m^{-T} \text{diag}(\omega_r^2) S_m^{-1} \tag{5.62}$$

where ω_i are the natural frequencies of the system and S_m is the matrix of modal vectors normalized with respect to the mass matrix. Substitution of these 'modal' expressions into Equation (5.59) yields

$$\begin{aligned}\alpha(\omega) &= \{S_m^{-T}[\text{diag}(\omega_r^2) - \omega^2 I]S_m^{-T}\}^{-1} \\ &= S_m\{\text{diag}[\omega_r^2 - \omega^2]^{-1}\}S_m^T\end{aligned} \tag{5.63}$$

Expression (5.63) can also be written in summation notation by considering the ijth element of $\alpha(\omega)$, recalling formula (2.6), and partitioning the matrix S_m into columns, denoted by $\mathbf{s}_r$. The vectors $\mathbf{s}_r$ are, of course, the eigenvectors of the matrix K normalized with respect to the mass matrix M. This yields

$$\alpha(\omega) = \sum_{r=1}^{n} (\omega_r^2 - \omega^2)^{-1} \mathbf{s}_r \mathbf{s}_r^T \tag{5.64}$$

The ijth element of the receptance matrix becomes

$$\alpha_{ij}(\omega) = \sum_{r=1}^{n} (\omega_r^2 - \omega^2)^{-1} [\mathbf{s}_r \mathbf{s}_r^T]_{ij} \tag{5.65}$$

where the matrix element $[\mathbf{s}_r \mathbf{s}_r^T]_{ij}$ is identified as the *modal constant* or *residue* for the rth mode, and the matrix $\mathbf{s}_r \mathbf{s}_r^T$ is called the *residue matrix*. Note that the right-hand side of Equation (5.65) can also be rationalized to form a single fraction consisting of the ratio of two polynomials in ω^2. Hence, $[\mathbf{s}_r \mathbf{s}_r^T]_{ij}$ can also be viewed as the matrix of constants in the partial fraction expansion of Equation (5.60).

Next, consider the same procedure applied to Equation (5.36) with nonzero damping. As always, consideration of damped systems results in two cases: those systems that decouple and those that do not.

First consider Equation (5.36) with damping such that $DM^{-1}K = KM^{-1}D$, so that the system decouples and the system eigenvectors are real. In this case the eigenvectors of the undamped system are also eigenvectors for the damped system, as was established in Section 3.5. The definition of the receptance matrix takes on a slightly different form to reflect the damping in the system. Under the additional assumption that the system is underdamped, i.e., that the matrix $4\tilde{K} - \tilde{D}^2$ is positive definite, the modal damping ratios ζ_r are all between 0 and 1. Equation (5.58) becomes

$$\mathbf{u} = (K + j\omega D - \omega^2 M)^{-1} \mathbf{f}_0 \tag{5.66}$$

Because the system decouples, matrix D can be written as

$$D = S_m^{-T} \operatorname{diag}(2\zeta_r \omega_r) S_m^{-1} \tag{5.67}$$

Substitution of Equations (5.61), (5.62), and (5.67) into Equation (5.66) yields

$$\mathbf{u} = S_m[\operatorname{diag}(\omega_r^2 + 2j\zeta_r\omega_r\omega - \omega^2)^{-1}]S_m^T \mathbf{f}_0 \tag{5.68}$$

This expression defines the complex receptance matrix given by

$$\alpha(\omega) = \sum_{r=1}^{n} (\omega_r^2 + 2j\zeta_r\omega_r\omega - \omega^2)^{-1}\mathbf{s}_r\mathbf{s}_r^T \tag{5.69}$$

Next, consider the general viscously damped case. In this case the eigenvectors $\mathbf{s}_r$ are complex and the receptance matrix is given (see Lancaster, 1966) as

$$\alpha(\omega) = \sum_{r=1}^{n} \left\{ \frac{\mathbf{s}_r\mathbf{s}_r^T}{j\omega - \lambda_r} + \frac{\mathbf{s}_r^*\mathbf{s}_r^{*T}}{j\omega - \lambda_r^*} \right\} \tag{5.70}$$

Here, the asterisk denotes the conjugate, the λ_r are the complex system eigenvalues, and the $\mathbf{s}_r$ are the system eigenvectors.

The expressions for the receptance matrix and the interpretation of an element of the receptance matrix given by Equation (5.60) form the background for modal testing. In addition, the receptance matrix forms a response model for the system. Considering the most general case [Equation (5.70)], the phenomenon of resonance is evident. In fact, if the real part of λ_r is small, $j\omega - \lambda_r$ is potentially small, and the response will be dominated by the associated mode $\mathbf{s}_r$. The receptance matrix is a generalization of the frequency response function of Section 1.5. In addition, like the transition matrix of Section 5.2, the receptance matrix maps the input of the system into the output of the system.

5.8 NUMERICAL SIMULATION IN MATLAB

This section extends Section 3.9 to include simulation of systems subject to an applied force. The method is the same as that described in Section 3.9, with the exception that the forcing function is now included in the equations of motion. All the codes mentioned in Section 3.9 have the ability numerically to integrate the equations of motion including both the effects of the initial conditions and the effects of any applied forces. Numerical simulation provides an alternative to computing the time response by modal methods, as done in Equation (5.45). The approach is to perform numerical integration following the material in Sections 1.10 and 3.9 with the state-space model. For any class of second-order systems the equations of motion can be written in the state-space form, as by Equation (5.1), subject to appropriate initial conditions on the position and velocity. While numerical solutions are a discrete time approximation, they are systematic and relatively easy to compute with modern high-level codes. The following example illustrates the procedure in MATLAB.

Example 5.8.1

Consider the system in physical coordinates defined by:

$$\begin{bmatrix} 5 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} \ddot{q}_1 \\ \ddot{q}_2 \end{bmatrix} + \begin{bmatrix} 3 & -0.5 \\ -0.5 & 0.5 \end{bmatrix}\begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{bmatrix} + \begin{bmatrix} 3 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} q_1 \\ q_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}\sin(4t),$$

$$\mathbf{q}(0) = \begin{bmatrix} 0 \\ 0.1 \end{bmatrix}, \qquad \dot{\mathbf{q}}(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

In order to use the Runge–Kutta numerical integration, first put the system into the state-space form. Computing the inverse of the mass matrix and defining the state vector $\mathbf{x}$ by

$$\mathbf{x} = \begin{bmatrix} \mathbf{q} \\ \dot{\mathbf{q}} \end{bmatrix} = \begin{bmatrix} q_1 & q_2 & \dot{q}_1 & \dot{q}_2 \end{bmatrix}^T$$

the state equations become

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -3/5 & 1/5 & -3/5 & 1/2 \\ 1 & -1 & 1/2 & -1/2 \end{bmatrix}\mathbf{x} + \begin{bmatrix} 0 \\ 0 \\ 1/5 \\ 1 \end{bmatrix}\sin 4t, \qquad \mathbf{x}(0) = \begin{bmatrix} 0 \\ 0.1 \\ 1 \\ 0 \end{bmatrix}$$

The steps to solve this numerically in MATLAB follow those of example 3.9.1, with the additional term for the forcing function. The corresponding m-file is

```
function v=f581(t,x)
M=[5 0; 0 5]; D=[3 -0.5;-0.5 0.5]; K=[3 -1;-1 1];
A=[zeros(2) eye(2);-inv(M)* K -inv(M)* D]; b=[0;0;0.2;1];
v=A*x+b*sin(4*t);
```

This function must be saved under the name `f581.m.` Once this is saved, the following is typed in the command window:

```
EDU>clear all
EDU>xo=[0;0.1;1;0];
EDU>ts=[0 50];
EDU>[t,x]=ode45('f581',ts,xo);
EDU>plot(t,x(:,1),t,x(:,2),'--'),title('x1,x2 versus time')
```

This returns the plot shown in Figure 5.1. Note that the command `x(:,1)` pulls off the record for $x_1(t)$ and the command ode45 calls a fifth-order Runge–Kutta program. The command `ts=[0 50];` tells the code to integrate from 0 to 50 time units (seconds in this case).

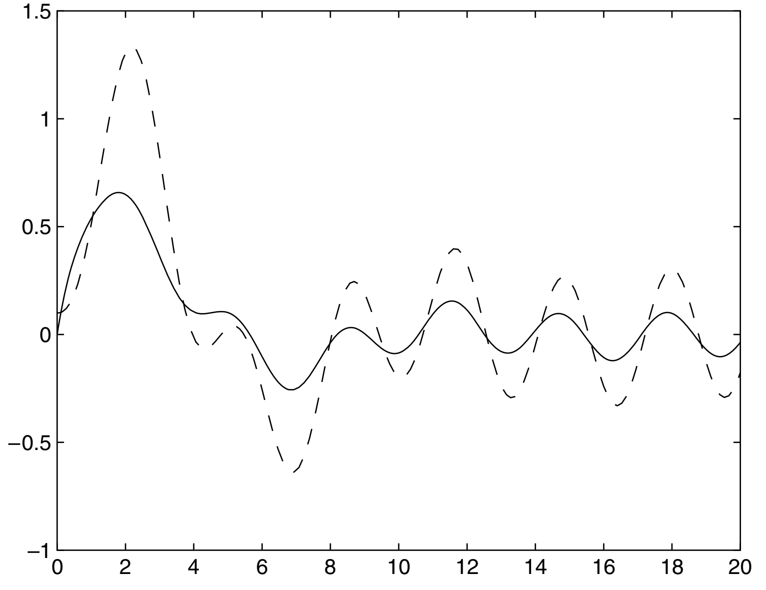

Figure 5.1 Displacement response to the initial conditions and forcing function of example 5.8.1

CHAPTER NOTES

The field of systems theory and control has advanced the idea of using matrix methods for solving large systems of differential equations (see Patel, Laub and Van Dooren, 1994). Thus, the material in Section 5.2 can be found in most introductory systems theory texts such as Chen (1998) or Kailath (1980). In addition, those texts contain material on modal decoupling of the state matrix, as covered in Section 5.3. Theoretical modal analysis (Section 5.3) is just a method of decoupling the equations of motion of a system into a set of simple-to-solve single-degree-of-freedom equations. This method is extended in Section 5.4 and generalized to equations that cannot be decoupled. For such systems, modal analysis of the solution is simply an expansion of the solution in terms of its eigenvectors. This material parallels the development of the free response in Section 3.5. The material of Section 5.6 on bounds is not widely used. However, it does provide some methodology for design work. The results presented in Section 5.6 are from Yae and Inman (1987). The material on frequency response methods presented in Section 5.7 is essential in understanding experimental modal analysis and testing and is detailed in Ewins (2000). Section 5.8 is a brief introduction to the important concept of numerical simulation of dynamic systems.

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PROBLEMS

5.1 Use the resolvent matrix to calculate the solution of

$$\ddot{x} + \dot{x} + 2x = \sin 3t$$

with zero initial conditions.

5.2 Calculate the transition matrix e^{At} for the system of problem 5.1.

5.3 Prove that $e^{-A} = (e^{A})^{-1}$ and show that $e^{A}e^{-A} = I$.

5.4 Compute e^{At}, where

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$$

5.5 Show that, if $\mathbf{z}(t) = ae^{-j\phi}e^{(\mu+j\omega)t}\mathbf{f}(t)$, where a and $\mathbf{f}(t)$ are real and $j^2 = -1$, then $\text{Re}(\mathbf{z}) = a\mathbf{f}(t)\,e^{\mu t}\cos(\omega t - \phi)$.

5.6 Consider problem 3.6. Let $\mathbf{f}(t) = [\mu(t) \;\; 0 \;\; 0]^T$, where $\mu(t)$ is the unit step function, and calculate the response of that system with $\mathbf{f}(t)$ as the applied force and zero initial conditions.

5.7 Let $\mathbf{f}(t) = [\sin(t) \;\; 0]^T$ in problem 3.10 and solve for $\mathbf{x}(t)$.

5.8 Calculate a bound on the forced response of the system given in problem 5.7. Which was easier to calculate, the bound or the actual response?

5.9 Calculate the receptance matrix for the system of example 5.6.1 with $\zeta = -1$.

5.10 Discuss the similarities between the receptance matrix, the transition matrix, and the resolvent matrix.

5.11 Using the definition of the matrix exponential, prove each of the following:

(a) $(e^{At})^{-1} = e^{-At}$;
(b) $e^0 = I$;
(c) $e^A e^B = e^{(A+B)}$, if $AB = BA$.

5.12 Develop the formulation for modal analysis of symmetrizable systems by applying the transformations of Section 4.9 to the procedure following Equation (5.23).
5.13 Using standard methods of differential equations, solve Equation (5.39) to obtain Equation (5.40).
5.14 Plot the response of the system in example 5.6.1 along with the bound indicated in Equation (5.54) for the case $\zeta = -1$.
5.15 Derive the solution of Equation (5.39) for the case $\zeta_i > 1$.
5.16 Show that $\mathbf{f}_0 e^{j\omega t}$ is periodic.
5.17 Compute the modal equations for the system described by

$$\begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix} \ddot{\mathbf{q}}(t) + \begin{bmatrix} 5 & -1 \\ -1 & 1 \end{bmatrix} \mathbf{q}(t) = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \sin 2t$$

subject to the initial conditions of zero initial velocity and an initial displacement of $\mathbf{x}(0) = [0 \quad 1]^T$ mm.
5.18 Repeat problem 5.17 for the same system with the damping matrix defined by $C = 0.1K$.
5.19 Derive the relationship between the transformations S and S_m.
5.20 Consider example problem 5.4.1 and compute the total response in physical coordinates.
5.21 Consider example problem 5.4.1 and plot the response in physical coordinates.
5.22 Consider the problem of example 5.4.1 and use the method of numerical integration discussed in Section 5.8 to solve and plot the solution. Compare your results to the analytical solution found in problem 5.21.
5.23 Consider the following undamped system:

$$\begin{bmatrix} 4 & 0 \\ 0 & 9 \end{bmatrix} \begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} + \begin{bmatrix} 30 & -5 \\ -5 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0.23500 \\ 2.97922 \end{bmatrix} \sin(2.756556t)$$

(a) Compute the natural frequencies and mode shapes and discuss whether or not the system experiences resonance.
(b) Compute the modal equations.
(c) Simulate the response numerically.

5.24 For the system of example 5.4.1, plot the frequency response function over a range of frequencies from 0 to 8 rad/s.
5.25 Compute and plot the frequency response function for the system of example 5.4.1 for the damping matrix having the value $D = \alpha K$ for several different values of α ranging from 0.1 to 1. Discuss your results. What happens to the peaks?