6
Design Considerations

6.1 INTRODUCTION

The word *design* means many different things to different people. Here, design is used to denote an educated method of choosing and adjusting the physical parameters of a vibrating system in order to obtain a more favorable response. The contents of this chapter are somewhat chronological in the sense that the topics covered first, such as vibration absorbers, are classical vibration design techniques, whereas the later sections, such as the one on control, represent more contemporary methods of design. A section on controls serves as an introduction to Chapter 7, which is devoted entirely to control methods. A section on damping treatments introduces a commonly used method of vibration suppression. The chapter ends with a section on model reduction, which is not a design method but a technique commonly used to provide reasonable sized models to help in design analysis.

6.2 ISOLATORS AND ABSORBERS

Isolation of a vibrating mass refers to designing the connection of a mass (machine part or structure) to ground in such a way as to reduce unwanted effects or disturbances through that connection. Vibration absorption, on the other hand, refers to adding an additional degree of freedom (spring and mass) to the structure to *absorb* the unwanted disturbance. The typical model used in vibration isolation design is the simple single-degree-of-freedom system of Figure 1.1(a) without damping, or Figure 1.4(a) with damping. The idea here is twofold. First, if a harmonic force is applied to the mass through movement of the ground (i.e., as the result of a nearby rotating machine, for instance), the values of c and k should be chosen to minimize the resulting response of the mass. The design *isolates* the mass from the effects of ground motion. The springs on an automobile serve this purpose.

A second use of the concept of isolation is that in which the mass represents the mass of a machine, causing an unwanted harmonic disturbance. In this case the values of m, c, and k are chosen so that the disturbance force passing through the spring and dashpot to ground is minimized. This isolates the ground from the effects of the machine. The motor mounts in an automobile are examples of this type of isolation.

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In either case, the details of the governing equations for the isolation problem consist of analyzing the steady state forced harmonic response of equations of form (1.17). For instance, if it is desired to isolate the mass of Figure 1.8 from the effects of a disturbance $\mathbf{F}_0 \sin(\omega t)$, then the magnification curves of Figure 1.9 indicate how to choose the damping ζ and the isolator frequency ω_n so that the amplitude of the resulting vibration is as small as possible. Curves similar to the magnification curves, called transmissibility curves, are usually used in isolation problems.

The ratio of the amplitude of the force transmitted through the connection between the ground and the mass to the amplitude of the driving force is called the *transmissibility*. For the system of Figure 1.8, the force transmitted to ground is transmitted through the spring, k, and the damper, c. From Equation (1.21), these forces at steady state are

$$F_k = kx_{ss}(t) = kX \sin(\omega t - \phi) \tag{6.1}$$

and

$$F_c = c\dot{x}_{ss}(t) = c\omega X \cos(\omega t - \phi) \tag{6.2}$$

Here, F_k and F_c denote the force in the spring and the force in the damper respectively, and X is the magnitude of the steady state response as given in Section 1.4. The magnitude of the transmitted force is the magnitude of the vector sum of these two forces, denoted by F_T, and is given by

$$F_T^2 = |kx_{ss} + c\dot{x}_{ss}|^2 = [(kX)^2 + (c\omega X)^2] \tag{6.3}$$

Thus, the magnitude of transmitted force becomes

$$F_T = kX\left[1 + \left(\frac{c\omega}{k}\right)^2\right]^{1/2} \tag{6.4}$$

The amplitude of the applied force is just F_0, so that the transmissibility ratio, denoted by TR, becomes

$$TR = \frac{F_T}{F_0} = \frac{\sqrt{1 + (2\zeta\omega/\omega_n)^2}}{\sqrt{[1 - (\omega/\omega_n)^2]^2 + [2\zeta\omega/\omega_n]^2}} \tag{6.5}$$

Plots of expression (6.5) versus the frequency ratio ω/ω_n for various values of ζ are called *transmissibility curves*. One such curve is illustrated in Figure 6.1. This curve indicates that, for values of $\omega/\omega_n > \sqrt{2}$ (that is, $TR < 1$), vibration isolation occurs, whereas for values of $\omega/\omega_n < \sqrt{2}$ ($TR > 1$) an amplification of vibration occurs. Of course, the largest increase in amplitude occurs at resonance.

If the physical parameters of a system are constrained such that isolation is not feasible, a vibration absorber may be included in the design. A vibration absorber consists of an attached second mass, spring, and damper, forming a two-degree-of-freedom system. The second spring–mass system is then 'tuned' to resonate and hence absorb all the vibrational energy of the system.

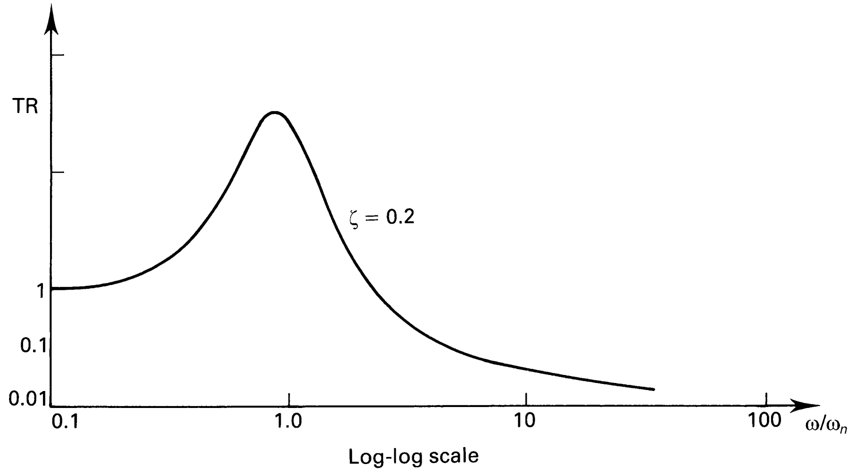

Figure 6.1 Transmissibility curve used in determining frequency values for vibration isolation.

The basic method of designing a vibration absorber is illustrated here by examining the simple case with no damping. To this end, consider the two-degree-of-freedom system of Figure 2.4 with $c_1 = c_2 = f_2 = 0$, $m_2 = m_a$, the absorber mass, $m_1 = m$, the primary mass, $k_1 = k$, the primary stiffness, and $k_2 = k_a$, the absorber spring constant. In addition, let $x_1 = x$, the displacement of the primary mass, and $x_2 = x_a$, the displacement of the absorber. Also, let the driving force $F_0 \sin(\omega t)$ be applied to the primary mass, m. The absorber is designed to the steady state response of this mass by choosing the values of m_a and k_a. Recall that the steady state response of a harmonically excited system is found by assuming a solution that is proportional to a harmonic term of the same frequency as the driving frequency.

From Equation (2.25) the equations of motion of the two-mass absorber system are

$$\begin{bmatrix} m & 0 \\ 0 & m_a \end{bmatrix}\begin{bmatrix} \ddot{x} \\ \ddot{x}_a \end{bmatrix} + \begin{bmatrix} k+k_a & -k_a \\ -k_a & k_a \end{bmatrix}\begin{bmatrix} x \\ x_a \end{bmatrix} = \begin{bmatrix} F_0 \\ 0 \end{bmatrix} \sin \omega t \tag{6.6}$$

Assuming that in the steady state the solution of Equation (6.6) will be of the form

$$\begin{bmatrix} x(t) \\ x_a(t) \end{bmatrix} = \begin{bmatrix} X \\ X_a \end{bmatrix} \sin \omega t \tag{6.7}$$

and substituting into Equation (6.6) yields

$$\begin{bmatrix} k+k_a-m\omega^2 & -k_a \\ -k_a & k_a - m_a\omega^2 \end{bmatrix}\begin{bmatrix} X \\ X_a \end{bmatrix} \sin \omega t = \begin{bmatrix} F_0 \\ 0 \end{bmatrix} \sin \omega t \tag{6.8}$$

Solving for the magnitudes $\mathbf{X}$ and $\mathbf{X}_a$ yields

$$\begin{bmatrix} X \\ X_a \end{bmatrix} = \frac{1}{(k+k_a-m\omega^2)(k_a-m_a\omega^2)-k_a^2}\begin{bmatrix} (k_a-m_a\omega^2) & k_a \\ k_a & (k+k_a-m\omega^2) \end{bmatrix}\begin{bmatrix} F_0 \\ 0 \end{bmatrix} \tag{6.9}$$

or

$$X = \frac{(k_a - m_a\omega^2)F_0}{(k + k_a - m\omega^2)(k_a - m_a\omega^2) - k_a^2} \tag{6.10}$$

and

$$X_a = \frac{k_a F_0}{(k + k_a - m\omega^2)(k_a - m_a\omega^2) - k_a^2} \tag{6.11}$$

As can be seen by examining Equation (6.10), if k_a and m_a are chosen such that $k_a - m_a\omega^2 = 0$, i.e., such that $\sqrt{k_a/m_a} = \omega$, then the magnitude of the steady state response of the primary m is zero, i.e., $X = 0$. Hence, if the added absorber mass, m_a, is 'tuned' to the driving frequency ω, then the amplitude of the steady state vibration of the primary mass, X, is zero and the absorber mass effectively absorbs the energy in the system.

The addition of damping into the absorber-mass system provides two more parameters to be adjusted for improving the response of the mass m. However, with damping, the magnitude X cannot be made exactly zero. The next section illustrates methods for choosing the design parameters to make X as small as possible in the damped case.

6.3 OPTIMIZATION METHODS

Optimization methods (see, for instance, Gill, Murray, and Wright, 1981) can be used to obtain the 'best' choice of the physical parameters m_a, c_a, and k_a in the design of a vibration absorber or, for that matter, any degree-of-freedom vibration problem (Vakakis and Paipetis, 1986). The basic optimization problem is described in the following and then applied to the damped vibration absorber problem mentioned in the preceding section.

The general form for standard nonlinear programming problems is to minimize some scalar function of the vector of design variables $\mathbf{y}$, denoted by $J(\mathbf{y})$, subject to p inequality constraints and q equality constraints, denoted by

$$g_s(\mathbf{y}) < 0, \qquad s = 1, 2, \ldots, p \tag{6.12}$$

$$h_r(\mathbf{y}) = 0, \qquad r = 1, 2, \ldots, q \tag{6.13}$$

respectively. The function $J(\mathbf{y})$ is referred to as the *objective function*, or *cost function*. The process is an extension of the constrained minimization problems studied in beginning calculus.

There are many methods available to solve such optimization problems. The intention of this section is not to present these various methods of design optimization but rather to introduce the use of optimization techniques as a vibration design method. The reader should consult one or more of the many texts on optimization for details of the various methods.

A common method for solving optimization problems with equality constraints is to use the method of *Lagrange multipliers*. This method defines a new vector $\theta = [\theta_1\ \theta_2 \cdots \theta_q]^T$ called the vector of Lagrange multipliers (in optimization literature they are sometimes denoted by λ_i), and the constraints are added directly to the objective function by using the scalar term $\theta^T\mathbf{h}$. The *new cost function* becomes $J' = J(\mathbf{y}) + \theta^T\mathbf{h}(\mathbf{y})$, which is then minimized as a function of y_i and θ_i. This is illustrated in the following example.

Example 6.3.1

Suppose it is desired to find the smallest value of the damping ratio ζ and the frequency ratio $r = \omega/\omega_n$ such that the transmissibility ratio is 0.1. The problem can be formulated as follows. Since TR = 0.1, then $TR^2 = 0.01$ or $(TR)^2 - 0.01 = 0$, which is the constraint $h_1(\mathbf{y})$ in this example. The vector $\mathbf{y}$ becomes $\mathbf{y} = [\zeta \quad r]^T$, and the vector θ is reduced to the scalar θ. The cost function J' then becomes

$$\begin{aligned} J' &= \zeta^2 + r^2 + \theta(TR^2 - 0.01) \\ &= \zeta^2 + r^2 + \theta[0.99 + 0.02r^2 - 0.01r^4 + 3.96\zeta^2 r^2] \end{aligned}$$

The necessary (but not sufficient) conditions for a minimum are that the first derivatives of the cost function with respect to the design variables must vanish. This yields

$$\begin{aligned} J'_\zeta &= 2\zeta + \theta(7.92\zeta r^2) = 0 \\ J'_r &= 2r + \theta[0.04r - 0.04r^3 + 7.92\zeta^2 r] = 0 \\ J'_\theta &= 0.99 + 0.02r^2 - 0.01r^4 + 3.96\zeta^2 r^2 = 0 \end{aligned}$$

where the subscripts of J' denote partial differentiation with respect to the given variable. These three nonlinear algebraic equations in the three unknowns ζ, r, and θ can be solved numerically to yield $\zeta = 0.037$, $r = 3.956$, and $\theta = -0.016$.

The question arises as to how to pick the objective function $J(\mathbf{y})$. The choice is arbitrary, but the function should be chosen to have a single global minimum. Hence, the choice of the quadratic form $(\zeta^2 + r^2)$ in the previous example. The following discussion on absorbers indicates how the choice of the cost function affects the result of the design optimization. The actual minimization process can follow several formulations; the results presented next follow Fox (1971).

Soom and Lee (1983) examined several possible choices of the cost function $J(\mathbf{y})$ for the absorber problem and provided a complete analysis of the absorber problem (for a two-degree-of-freedom system) given by

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_a \end{bmatrix}\begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_a \end{bmatrix} + \begin{bmatrix} c_1 + c_a & -c_a \\ -c_a & c_a \end{bmatrix}\begin{bmatrix} \dot{x}_1 \\ \dot{x}_a \end{bmatrix} + \begin{bmatrix} k_1 + k_a & -k_a \\ -k_a & k_a \end{bmatrix}\begin{bmatrix} x_1 \\ x_a \end{bmatrix} = \begin{bmatrix} f \\ 0 \end{bmatrix}\cos\omega t \tag{6.14}$$

These equations are first nondimensionalized by defining the following new variables and constants:

$$\omega_1 = \sqrt{\frac{k_1}{m_1}}, \qquad \zeta_1 = \frac{c_1}{2\sqrt{m_1 k_1}}$$
$$\omega = \frac{\omega}{\omega_1}, \qquad \zeta_2 = \frac{c_a}{2\sqrt{m_1 k_1}}$$
$$\tau = \omega_1, \qquad P = \frac{f}{k_1 L}, \text{ where } L \text{ is the static deflection of } x_1$$
$$\mu = \frac{m_a}{m_1}, \qquad z_1 = \frac{x_1}{L}$$
$$k = \frac{k_a}{k_1}, \qquad z_2 = \frac{x_a}{L}$$

Substitution of these into Equation (6.14) and dividing by $k_1 L$ yields the dimensionless equations

$$\begin{bmatrix} 1 & 0 \\ 0 & \mu \end{bmatrix} \begin{bmatrix} \ddot{z}_1 \\ \ddot{z}_2 \end{bmatrix} + \begin{bmatrix} 2(\zeta_1 + \zeta_2) & -2\zeta_2 \\ -2\zeta_2 & 2\zeta_2 \end{bmatrix} \begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \end{bmatrix} + \begin{bmatrix} 1+k & -k \\ -k & k \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} P \\ 0 \end{bmatrix} \cos \omega\tau \tag{6.15}$$

where the overdots now indicate differentiation with respect to τ. As before, the steady state responses of the two masses are assumed to be of the form

$$\begin{aligned} z_1 &= |A_1| \cos(\omega\tau + \phi_1) \\ z_2 &= |A_2| \cos(\omega\tau + \phi_2) \end{aligned} \tag{6.16}$$

Substitution of Equations (6.16) into Equation (6.15) and solving for the amplitudes $|A_1|$ and $|A_2|$ yields

$$|A_2| = \sqrt{(a/q)^2 + (b/q)^2}\ |A_1| \tag{6.17}$$

$$|A_1| = \frac{P}{\sqrt{(1 - \omega^2 - r/q)^2 + (2\zeta_1\omega + s/q)^2}} \tag{6.18}$$

where the constants a, b, q, r, and s are defined by

$$\begin{aligned} a &= k^2 + 4\zeta_2^2\omega^2 - \mu k \omega^2 \\ b &= -2\zeta_2 \mu \omega^3 \\ q &= (k - \mu\omega^2)^2 + 4\zeta_2^2\omega^2 \\ r &= \mu k^2 \omega^2 - \mu^2 k \omega^4 + 4\zeta_2^2 \omega^4 \\ s &= 2\zeta_2 \mu^2 \omega^5 \end{aligned}$$

Note that Equations (6.17) and (6.18) are similar in form to Equations (6.10) and (6.11) for the undamped case. However, the tuning condition is no longer obvious, and there are many possible design choices to make. This marks the difference between an absorber with damping and one without. The damped case is, of course, a much more realistic model of the absorber dynamics.

Optimization methods are used to make the best design choice among all the physical parameters. The optimization is carried out using the design variable defined by the design vector. Here, α is the tuning condition defined by

$$\alpha = \sqrt{\frac{k}{\mu}}$$

and ζ_2' is a damping ratio defined by

$$\zeta_2' = \frac{\zeta_2}{\sqrt{\mu k}}$$

The quantity ζ_2' is the damping ratio of the 'added-on' absorber system of mass m_a. The tuning condition α is the ratio of the two undamped natural frequencies of the two masses.

The designer has the choice of making up objective functions. In this sense, the optimization produces an arbitrary best design. Choosing the objective function is the art of optimal design. However, several cost or objective functions can be used, and the results of each optimization compared. Soom and Lee (1983) considered several different objective functions:

J_1 = the maximum value of $|A_1|$, the magnitude of the displacement response in the frequency domain;
J_2 = $\sum(|A_1| - 1)^2$ for frequencies where $|A_1| > 1$ and where the sum runs over a number of discrete points on the displacement response curves of mass m_1;
J_3 = maximum $(\omega|A_1|)$, the maximum velocity of m_1;
J_4 = $\sum |A_1|^2$, the mean squared displacement response;
J_5 = $\sum(\omega|A_1|)^2$, the mean squared velocity response.

These objective functions were all formed by taking 100 equally spaced points in the frequency range from $\omega = 0$ to $\omega = 2$. The only constraints imposed were that the stiffness and damping coefficients be positive.

Solutions to the various optimizations yields the following interesting design conclusions:

1. From minimizing J_1, the plot of J_1 versus ζ_1 for various mass ratios is given in Figure 6.2 and shows that one would not consider using a dynamic absorber for a system with a damping ratio much greater than $\zeta_1 = 0.2$. The plots clearly show that not much reduction in magnitude can be expected for systems with large damping in the main system.
2. For large values of damping, $\zeta_1 = 0.3$, the different objective functions lead to different amounts of damping in the absorber mass, ζ_2, and the tuning ratio, α. Thus, the choice of the objective function changes the optimum point. This is illustrated in Figure 6.3.
3. The peak, or maximum value, of $z_1(t)$ at resonance also varies somewhat, depending on the choice of the cost function. The lowest reduction in amplitude occurs with objective function J_1, as expected, which is 30% lower than the value for J_4.

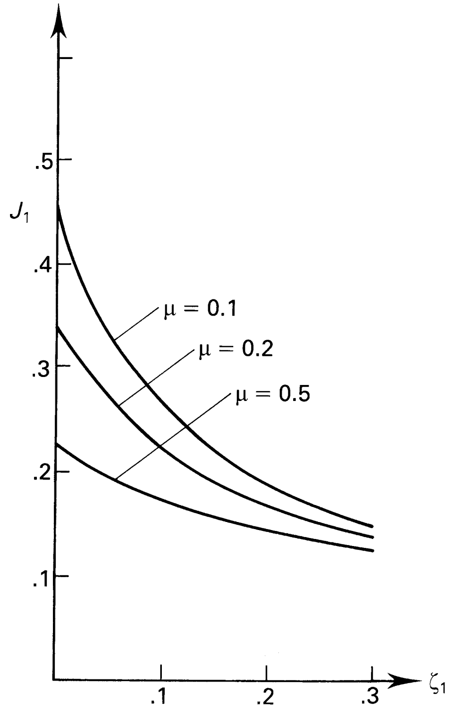

Figure 6.2 Plot of the cost function J_1, versus the damping ratio ζ_1 for various mass ratios.

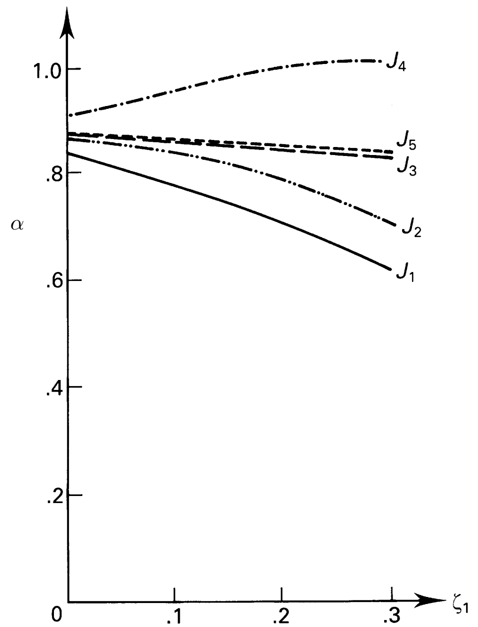

Figure 6.3 Tuning parameter α versus the first mode damping ratio ζ_1 for each cost function J, indicating a broad range of optimal values of α.

It was concluded from this study that, while optimal design can certainly improve the performance of a device, a certain amount of ambiguity exists in an optimal design based on the choice of the cost function. Thus, the cost function must be chosen with some understanding about the design objectives as well as physical insight.

6.4 DAMPING DESIGN

This section illustrates a method of adjusting the individual mass, damping, and stiffness parameters of a structure in order to produce a desired damping ratio. Often in the design of systems, damping is introduced to achieve a reduced level of vibrations, or to perform *vibration suppression*. Consider a symmetric system of the form

$$M\ddot{\mathbf{x}} + D\dot{\mathbf{x}} + K\mathbf{x} = \mathbf{0} \tag{6.19}$$

where M, D, and K are the usual symmetric, positive definite mass, damping, and stiffness matrices, to be adjusted so that the modal damping ratios, ζ_i, have desired values. This in turn provides insight into how to adjust or design the individual elements m_i, c_i, and k_i such that the desired damping ratios are achieved.

Often in the design of a mechanical part, the damping in the structure is specified in terms of either a value for the loss factor or a percentage of critical damping, i.e., the damping ratio. This is mainly true because these are easily understood concepts for a single-degree-of-freedom model of a system. However, in many cases, of course, the behavior of a given structure may not be satisfactorily modeled by a single modal parameter. Hence, the question of how to interpret the damping ratio for a multiple-degree-of-freedom system such as the symmetric positive definite system of Equation (6.19) arises.

An n-degree-of-freedom system has n damping ratios, ζ_i. These damping ratios are, in fact, defined by Equation (5.40) for the normal mode case (i.e., under the assumption that $DM^{-1}K$ is symmetric). Recall that, if the equations of motion decouple, then each mode has a damping ratio ζ_i defined by

$$\zeta_i = \frac{\lambda_i(D)}{2\omega_i} \tag{6.20}$$

where ω_i is the ith undamped natural frequency of the system and $\lambda_i(D)$ denotes the ith eigenvalue of matrix D.

To formalize this definition and to examine the nonnormal mode case ($DM^{-1}K \neq KM^{-1}D$), the *damping ratio matrix*, denoted by Z, is defined in terms of the critical damping matrix D_{cr} of Section 3.6. The damping ratio matrix is defined by

$$Z = D_{cr}^{-1/2}\tilde{D}D_{cr}^{-1/2} \tag{6.21}$$

where $\tilde{D}$ is the mass normalized damping matrix of the structure. Furthermore, define the matrix Z' to be the diagonal matrix of eigenvalues of matrix Z, i.e.,

$$Z' = \text{diag}\,[\lambda_i(Z)] = \text{diag}\,[\zeta_i^*] \tag{6.22}$$

Here, the ζ_i^* are damping ratios in that, if $0 < \zeta_i^* < 1$, the system is underdamped. Note, of course, that, if $DM^{-1}K = KM^{-1}D$, then $Z = Z'$.

By following the definitions of underdamped and critically damped systems of Section 3.6, it can easily be shown that the definiteness of the matrix $I - Z'$ determines whether a given system oscillates.

Example 6.4.1

As an example, consider Equation (6.19) with the following numerical values for the coefficient matrices:

$$M = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}, \qquad D = \begin{bmatrix} 6 & -\sqrt{2} \\ -\sqrt{2} & 1 \end{bmatrix}, \qquad K = \begin{bmatrix} 10 & -\sqrt{2} \\ -\sqrt{2} & 1 \end{bmatrix}$$

In this case, D_{cr} is calculated to be

$$D_{cr} = 2\tilde{K}^{1/2} = \begin{bmatrix} 4.4272 & -0.6325 \\ -0.6325 & 1.8974 \end{bmatrix}$$

where $\tilde{K} = M^{-1/2}KM^{-1/2}$.

From Equation (6.21) the damping ratio matrix becomes

$$Z = \begin{bmatrix} 0.3592 & -0.2205 \\ -0.2205 & 0.4660 \end{bmatrix}$$

It is clear that the matrix $[I - Z]$ is positive definite, so that each mode in this case should be underdamped. That is

$$(I - Z) = \begin{bmatrix} 0.3592 & 0.2205 \\ 0.2205 & 0.5340 \end{bmatrix}$$

so that the principle minors become $0.3592 > 0$ and $\det(I - Z) = 0.1432 > 0$. Hence, the matrix $[I - Z]$ is positive definite.

Calculating the eigenvalues of the matrix Z yields $\lambda_1(Z) = 0.7906$ and $\lambda_2(Z) = 0.3162$, so that $0 < \lambda_1(Z) < 1$ and $0 < \lambda_2(Z) < 1$, again predicting that the system is underdamped in each mode, since each ζ_i^* is between 0 and 1.

To illustrate the validity of these results for this example, the latent roots of the system can be calculated. They are

$$\lambda_{1,2} = -0.337 \pm 0.8326\,j, \qquad \lambda_{3,4} = -1.66 \pm 1.481\,j$$

where $j = \sqrt{-1}$. Thus, each mode is, in fact, underdamped as predicted by both the damping ratio matrix Z and the modal damping ratio matrix Z'.

It would be a useful design technique to be able to use this defined damping ratio matrix to assign damping ratios to each mode and back-calculate from the matrix Z' to obtain the required damping matrix D. Unfortunately, although the eigenvalues of matrix Z' specify the qualitative behavior of the system, they do not correspond to the actual modal damping ratios unless the matrix $DM^{-1}K$ is symmetric. However, if the damping is proportional, then Equation (6.21) can be used to calculate the desired damping matrix in terms of the specified damping ratios, i.e.,

$$\tilde{D} = D_{cr}^{1/2} Z D_{cr}^{1/2}$$

This damping matrix would then yield a system with modal damping ratios exactly as specified.

This section is a prelude to active control where one specifies the desired eigenvalues of a system (i.e., damping ratios and natural frequencies) and then computes a control law to achieve these values. The pole placement method introduced in Section 6.6 is such a method. The hardware concerns for achieving the desired damping rates are discussed in Nashif, Jones, and Henderson (1985).

6.5 DESIGN SENSITIVITY AND REDESIGN

Design sensitivity analysis usually refers to the study of the effect of parameter changes on the result of an optimization procedure or an eigenvalue–eigenvector computation. For instance, in the optimization procedure presented in Section 6.3, the nonabsorber damping ratio ζ_1 was not included as a parameter in the optimization. How the resulting optimum changes as ζ_1 changes is the topic of sensitivity analysis for the absorber problem. The eigenvalue and eigenvector perturbation analysis of Section 3.7 is an example of design sensitivity for the eigenvalue problem. This can, on the other hand, be interpreted as the *redesign* problem, which poses the question as to how much the eigenvalue and eigenvector solution changes as a specified physical parameter changes because of some other design process. In particular, if a design change causes a system parameter to change, the eigensolution can be computed without having to recalculate the entire eigenvalue/eigenvector set. This is also referred to as a *reanalysis* procedure and sometimes falls under the heading of *structural modification*. These methods are all fundamentally similar to the perturbation methods introduced in Section 3.7. This section develops the equations for discussing the sensitivity of natural frequencies and mode shapes for conservative systems.

The motivation for studying such methods comes from examining the large-order dynamical systems often used in current vibration technology. Making changes in large systems is part of the design process. However, large amounts of computer time are required to find the solution of the redesigned system. It makes sense, then, to develop efficient methods to update existing solutions when small design changes are made in order to avoid a complete reanalysis. In addition, this approach can provide insight into the design process.

Several approaches are available for performing a sensitivity analysis. The one presented here is based on parameterizing the eigenvalue problem. Consider a conservative n-degree-of-freedom system defined by

$$M(\alpha)\ddot{\mathbf{q}}(t) + K(\alpha)\mathbf{q}(t) = \mathbf{0} \tag{6.23}$$

where the dependence of the coefficient matrices on the design parameter α is indicated. The parameter α is considered to represent a change in the matrix M and/or the matrix K. The related eigenvalue problem is

$$M^{-1}(\alpha)K(\alpha)\mathbf{u}_i(\alpha) = \lambda_i(\alpha)\mathbf{u}_i(\alpha) \tag{6.24}$$

Here, the eigenvalue $\lambda_i(\alpha)$ and the eigenvector $\mathbf{u}_i(\alpha)$ will also depend on the parameter α. The mathematical dependence is discussed in detail by Whitesell (1980). It is assumed that the dependence is such that M, K, λ_i, and $\mathbf{u}_i$ are all twice differentiable with respect to the parameter α.

Proceeding, if $\mathbf{u}_i$ is normalized with respect to the mass matrix, differentiation of Equation (6.24) with respect to the parameter α yields

$$\frac{d}{d\alpha}(\lambda_i) = \mathbf{u}_i^T \left[\frac{d}{d\alpha}(K) - \lambda_i \frac{d}{d\alpha}(M) \right] \mathbf{u}_i \tag{6.25}$$

Here, the dependence of α has been suppressed for notational convenience. The second derivative of λ_i can also be calculated as

$$\begin{aligned}\frac{d^2}{d\alpha^2}\lambda_i = {} & 2\mathbf{u}'^T_i \left[\frac{d}{d\alpha}(K) - \lambda_i \frac{d}{d\alpha}(M) \right] \mathbf{u}'_i \\ & + \mathbf{u}_i^T \left[\frac{d^2}{d\alpha^2}(K) - \frac{d}{d\alpha}(\lambda_i)\frac{d}{d\alpha}(M) - \lambda_i \frac{d^2}{d\alpha^2}(M) \right] \mathbf{u}_i\end{aligned} \tag{6.26}$$

The notation $\mathbf{u}'$ denotes the derivative of the eigenvector with respect to α. The expression for the second derivative of λ_i requires the existence and computation of the derivative of the corresponding eigenvector. For the special case where M is a constant, and with some manipulation (see Whitesell, 1980), the eigenvector derivative can be calculated from the related problem for the eigenvector $\mathbf{v}_i$ from the formula

$$\frac{d}{d\alpha}(\mathbf{v}_i) = \sum_{k=1}^{n} c_k(\mathrm{i}, \alpha)\mathbf{v}_k \tag{6.27}$$

where the vectors $\mathbf{v}_k$ are related to $\mathbf{u}_k$ by the mass transformation $\mathbf{v}_k = M^{1/2}\mathbf{u}_k$. The coefficients $c_k(\mathrm{i}, \alpha)$ in this expansion are given by

$$c_k(\mathrm{i}, \alpha) = \begin{cases} 0 & i = k \\ \dfrac{1}{\lambda_i - \lambda_k}\mathbf{u}_k^T \dfrac{dA}{d\alpha}\mathbf{u}_i, & i \neq k \end{cases} \tag{6.28}$$

where the matrix A is the symmetric matrix $M^{-1/2}KM^{-1/2}$ depending on α.

Equations (6.25) and (6.27) yield the sensitivity of the eigenvalues and eigenvectors of a conservative system to changes in the stiffness matrix. More general and computationally efficient methods for computing these sensitivities are available in the literature. Adhikari and Friswell (2001) give formulae for damped systems and reference to additional methods.

Example 6.5.1

Consider the system discussed previously in example 3.3.2. Here, take $M = I$, and $\tilde{K}$ becomes

$$\tilde{K} = \begin{bmatrix} 3 & -1 \\ -1 & 3 \end{bmatrix} = K$$

The eigenvalues of the matrix are $\lambda_{1,2} = 2, 4$, and the normalized eigenvectors are $\mathbf{u}_1 = \mathbf{v}_1 = (1/\sqrt{2})\begin{bmatrix}1 & 1\end{bmatrix}^T$ and $\mathbf{u}_2 = \mathbf{v}_2 = (1/\sqrt{2})\begin{bmatrix}-1 & 1\end{bmatrix}^T$. It is desired to compute the sensitivity of the natural frequencies and mode shapes of this system as a result of a parameter change in the stiffness of the spring attached to ground. To this end, suppose the new design results in a new stiffness matrix of the form

$$K(\alpha) = \begin{bmatrix} 3+\alpha & -1 \\ -1 & 3 \end{bmatrix}$$

Then

$$\frac{d}{d\alpha}(M) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad \frac{d}{d\alpha}(K) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

Following Equations (6.25) and (6.27), the derivatives of the eigenvalues and eigenvectors become

$$\frac{d\lambda_1}{d\alpha} = 0.5, \qquad \frac{d\lambda_2}{d\alpha} = 0.5, \qquad \frac{d\mathbf{u}_1}{d\alpha} = \frac{1}{4\sqrt{2}}\begin{bmatrix} -1 \\ 1 \end{bmatrix}, \qquad \frac{d\mathbf{u}_2}{d\alpha} = \frac{-1}{4\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

These quantities are an indication of the sensitivity of the eigensolution to changes in the matrix K. To see this, substitute the preceding expressions into the expansions for $\lambda(\alpha)$ and $\mathbf{u}_i(\alpha)$ given by Equations (3.98) and (3.99). This yields

$$\lambda_1(\alpha) = 2 + 0.5\alpha, \qquad \lambda_2(\alpha) = 4 + 0.5\alpha$$
$$\mathbf{u}_1(\alpha) = 0.707\begin{bmatrix} 1 \\ 1 \end{bmatrix} + 0.177\alpha\begin{bmatrix} -1 \\ 1 \end{bmatrix}, \qquad \mathbf{u}_2(\alpha) = 0.707\begin{bmatrix} -1 \\ 1 \end{bmatrix} - 0.177\alpha\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

This last set of expressions allows the eigenvalues and eigenvectors to be evaluated for any given parameter change α without having to resolve the eigenvalue problem. These formulae constitute an approximate reanalysis of the system.

It is interesting to note this sensitivity in terms of a percentage. Define the percentage change in λ_1 by

$$\frac{\lambda_1(\alpha) - \lambda_1}{\lambda_1} 100\% = \frac{(2 + 0.5\alpha) - 2}{2} 100\% = (25\%)\alpha$$

If the change in the system is small, say $\alpha = 0.1$, then the eigenvalue λ_1 changes by only 2.5%, and the eigenvalue λ_2 changes by 1.25%. On the other hand, the change in the elements of the eigenvector $\boldsymbol{u}_2$ is 2.5%. Hence, in this case the eigenvector is more sensitive to parameter changes than the eigenvalue is.

By computing higher-order derivatives of λ_i and $\mathbf{u}_i$, more terms of the expansion can be used, and greater accuracy in predicting the eigensolution of the new system results. By using the appropriate matrix computations, the subsequent evaluations of the eigenvalues and eigenvectors as the design is modified can be carried out with substantially less computational effort (reportedly of the order of n^2 multiplications). The sort of calculation provided by eigenvalue and eigenvector derivatives can provide an indication of how changes to an initial design will affect the response of the system. In the example, the shift in value of the first spring is translated into a percentage change in the eigenvalues and hence in the natural frequencies. If the design of the system is concerned with avoiding resonance, then knowing how the frequencies shift with stiffness is critical.

6.6 PASSIVE AND ACTIVE CONTROL

In the redesign approach discussed in the previous section, the added structural modification α can be thought of as a passive control. If α represents added stiffness chosen to improve the vibrational response of the system, then it can be thought of as a passive control procedure. As mentioned in Section 1.8, passive control is distinguished from active control by the use of added power or energy in the form of an actuator, required in active control.

The material on isolators and absorbers of Section 6.2 represents two possible methods of passive control. Indeed, the most common passive control device is the vibration absorber. Much of the other work in passive control consists of added layers of damping material applied to various structures to increase the damping ratios of troublesome modes. Adding mass and changing stiffness values are also methods of passive control used to adjust a frequency away from resonance. Damping treatments increase the rate of decay of vibrations, so they are often more popular for vibration suppression.

Active control methods have been introduced in Sections 1.8, 2.3, and 4.10. Here we examine active control as a design method for improving the response of a vibrating system. This section introduces the method of *eigenvalue placement* (often called *pole placement*), which is useful in improving the free response of a vibrating system by shifting natural frequencies and damping ratios to desired values. The method of Section 6.4 is a primitive version of placing the eigenvalues by adjusting the damping matrix. The next chapter is devoted to formalizing and expanding this method (Section 7.3), as well as introducing some of the other techniques of control theory.

There are many different methods of approaching the eigenvalue placement problem. Indeed, it is the topic of ongoing research. The approach taken here is simple. The characteristic equation of the structure is written. Then a feedback law is introduced with undetermined gain coefficients of the form given by Equations (4.24) through (4.26). The characteristic equation of the closed-loop system is then written and compared with the characteristic equation of the open-loop system. Equating coefficients of the powers of λ in the two characteristic equations yields algebraic equations in the gain parameters, which are then solved. This yields the control law, which causes the system to have the desired eigenvalues. The procedure is illustrated in the following example.

Example 6.6.1

Consider the undamped conservative system of example 2.4.4 with $M = I$, $D = 0$, $k_1 = 2$, and $k_2 = 1$. The characteristic equation of the system becomes

$$\lambda^2 - 4\lambda + 2 = 0$$

This has roots $\lambda_1 = 2 - \sqrt{2}$ and $\lambda_2 = 2 + \sqrt{2}$. The natural frequencies of the system are then $\sqrt{2 - \sqrt{2}}$ and $\sqrt{2 + \sqrt{2}}$. Suppose now that it is desired to raise the natural frequencies of this system to be $\sqrt{2}$ and $\sqrt{3}$ respectively. Furthermore, assume that the values of k_i and m_i cannot be adjusted, i.e., that passive control is not a design option in this case.

First, consider the control and observation matrices of Section 4.10 and the solution to Problem 4.7. The obvious choice would be to measure the positions $q_1(t)$ and $q_2(t)$, so that $C_v = 0$ and $C_p = I$, and apply forces proportional to their displacements, so that

$$G_f = \begin{bmatrix} g_1 & 0 \\ 0 & g_2 \end{bmatrix}$$

with the actuators placed at x_1 and x_2 respectively. In this case, the matrix B_f becomes $B_f = I$. Then, the closed-loop system of Equation (4.27) has the characteristic equation

$$\lambda^2 - (4 + g_1 + g_2)\lambda + 2 + g_1 + 3g_2 + g_1 g_2 = 0 \qquad (6.29)$$

If it is desired that the natural frequencies of the closed-loop system be $\sqrt{2}$ and $\sqrt{3}$, then the eigenvalues must be changed to 2 and 3, which means the desired characteristic equation is

$$(\lambda - 3)(\lambda - 2) = \lambda^2 - 5\lambda + 6 = 0 \qquad (6.30)$$

By comparing the coefficients of λ and λ^0 (constant) terms of Equations (6.29) and (6.30), it can be seen that the gains g_1 and g_2 must satisfy

$$5 = (4 + g_1 + g_2)$$
$$6 = 2 + g_1 + 3g_2 + g_1 g_2$$

which has no real solutions.

From Equation (6.29) it is apparent that, in order to achieve the goal of placing the eigenvalues, and hence the natural frequencies, the gains must appear in some different order in the coefficients of Equation (6.29). This condition can be met by reexamining the matrix B_f. In fact, if B_f is chosen to be

$$B_f = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}$$

the characteristic equation for the closed-loop system becomes

$$\lambda^2 - (4 + g_2)\lambda + 2 + 3g_2 + g_1 = 0 \qquad (6.31)$$

Comparison of the coefficients of λ in Equations (6.30) and (6.31) yields values for the gains of $g_1 = 1$ and $g_2 = 1$.

The eigenvalues with these gains can be easily computed as a check to see that the scheme works. They are in fact $\lambda = 2$ and $\lambda = 3$, resulting in the desired natural frequencies.

As illustrated by the preceding example, the procedure is easy to calculate but does not always yield real values or even realistic values of the gains. The way in which G_f, B_f, and C_p are chosen and, in fact, whether or not such matrices even exist are topics of the next chapter. Note that the ability to choose these matrices is the result of the use of feedback and illustrates the versatility gained by using active control as against passive control. In passive control, g_1 and g_2 have to correspond to changes in mass or stiffness. In active control,

g_1 and g_2 are often electronic settings and hence are easily adjustable within certain bounds (but at other costs).

The use of pole placement assumes that the designer understands, or knows, what eigenvalues are desirable. This knowledge comes from realizing the effect that damping ratios and frequencies, and hence the eigenvalues, have on the system response. Often these are interpreted from, or even stated in terms of, design specifications. This is the topic of the next section.

6.7 DESIGN SPECIFICATIONS

The actual design of a mechanism starts and ends with a list of performance objectives or criteria. These qualitative criteria are eventually stated in terms of quantitative design specifications. Sample specifications form the topic of this section. Three performance criteria are considered in this section: speed of response, relative stability, and resonance.

The *speed of response* addresses the length of time required before steady state is reached. In classical control this is measured in terms of rise time, settling time, and bandwidth, as discussed in Section 1.4. In vibration analysis, speed of response is measured in terms of a decay rate or logarithmic decrement. Speed of response essentially indicates the length of time for which a structure or machine experiences transient vibrations. Hence, it is the time elapsed before the steady state response dominates. If just a single output is of concern, then the definitions of these quantities for multiple-degree-of-freedom systems are similar to those for the single-degree-of-freedom systems of Chapter 1.

For instance, for an n-degree-of-freedom system with position vector $\mathbf{q} = [q_1(t) \quad q_2(t) \ldots q_n(t)]^T$, if one force is applied, say at position m_1, and one displacement is of concern, say $q_8(t)$, then specifications for the speed of response of $q_8(t)$ can be defined as follows. The *settling time* is the time required for the response $q_8(t)$ to remain within $\pm\alpha$ percent of the steady state value of $q_8(t)$. Here, α is usually 2, 3, or 5. The *rise time* is the time required for the response $q_8(t)$ to go from 10 to 90% of its steady state value. The *log decrement* discussed in Equation (1.35) can be used as a measure of the decay rate of the system. All these specifications pertain to the transient response of a single-input, single-output (SISO) configuration.

On the other hand, if interest is in the total response of the system, i.e., the vector $\mathbf{q}$, then the response bounds of Section 5.6 yield a method of quantifying the decay rate for the system. In particular, the constant β, called a decay rate, may be specified such that

$$\|\mathbf{q}(t)\| < M\mathrm{e}^{-\beta t}$$

is satisfied for all $t > 0$. This can also be specified in terms of the *time constant* defined by the time, t, required for $\beta t = 1$. Thus, the time constant is $t = 1/\beta$.

Example 6.7.1

Consider the system of example 5.2.1. The response norm of the position is the first component of the vector $\mathbf{x}(t)$ so that $q(t) = (1 - e^{-t})e^{-t}$ and its norm is $|e^{-t} - e^{-2t}| < |e^{-t}| = e^{-t}$. Hence $\beta = 1$, and the decay rate is also 1.

Some situations may demand that the relative stability of a system be quantified. In particular, requiring that a system be designed to be stable or asymptotically stable may not be enough. This is especially true if some of the parameters in the system may change over a period of time or change owing to manufacturing tolerances or if the system is under active control. Often the concept of a stability margin is used to quantify relative stability.

In Chapter 4 several systems are illustrated that can become unstable as one or more parameters in the system change. For systems in which a single parameter can be used to characterize the stability behavior of the system, the *stability margin*, denoted by *sm*, of the system can be defined as the ratio of the maximum stable value of the parameter to the actual value for a given design configuration. The following example illustrates this concept.

Example 6.7.2

Consider the system defined in example 4.6.1 with $\gamma = 1, c_1 = 6$, and $c_2 = 2$ and calculate the stability margin of the system as the parameter changes. Here, η is being considered as a design parameter. As the design parameter η increases, the system approaches an unstable state. Suppose the operating value of η, denoted by η_{op}, is 0.1. Then, the stiffness matrix becomes semidefinite for $\eta = 1$ and indefinite for $\eta > 1$, and the maximum stable value of η is $\eta_{max} = 1$. Hence, the stability margin is

$$sm = \frac{\eta_{max}}{\eta_{op}} = \frac{1}{0.1} = 10$$

If the design of the structure is such that $\eta_{op} = 0.5$, then $sm = 2$. Thus, all other factors being equal, the design with $\eta_{op} = 0.1$ is 'more stable' than the same design with $\eta_{op} = 0.5$, because $\eta_{op} = 0.1$ has a larger stability margin.

The resonance properties, or modal properties, of a system are obvious design criteria in the sense that in most circumstances resonance is to be avoided. The natural frequencies, mode shapes, and modal damping ratios are often specified in design work. Methods of designing a system to have particular modal properties have been discussed briefly in this chapter in terms of passive and active control. Since these specifications can be related to the eigenvalue problem of the system, the question of designing a system to have specified modal properties is answered by the pole placement methods and eigenstructure assignment methods of control theory discussed in Section 7.3.

6.8 MODEL REDUCTION

A difficulty with many design and control methods is that they work best for systems with a small number of degrees of freedom. Unfortunately, many interesting problems have a large number of degrees of freedom. One approach to this dilemma is to reduce the size of the original model by essentially removing those parts of the model that affect its dynamic response of interest the least. This process is called *model reduction*, or *reduced-order modeling.*

Quite often the mass matrix of a system may be singular or nearly singular owing to some elements being much smaller than others. In fact, in the case of finite element modeling (discussed in Section 13.3), the mass matrix may contain zeros along a portion of the diagonal (called an *inconsistent mass matrix*). Coordinates associated with zero, or relatively small mass, are likely candidates for being removed from the model.

Another set of coordinates that are likely choices for removal from the model are those that do not respond when the structure is excited. Stated another way, some coordinates may have more significant responses than others. The distinction between significant and insignificant coordinates leads to a convenient formulation of the model reduction problem due to Guyan (1965).

Consider the undamped forced vibration problem of Equation (5.22) and partition the mass and stiffness matrices according to significant displacements, denoted by $\mathbf{q}_1$, and insignificant displacements, denoted by $\mathbf{q}_2$. This yields

$$\begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix}\begin{bmatrix} \ddot{\mathbf{q}}_1 \\ \ddot{\mathbf{q}}_2 \end{bmatrix} + \begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix}\begin{bmatrix} \mathbf{q}_1 \\ \mathbf{q}_2 \end{bmatrix} + \begin{bmatrix} \mathbf{f}_1 \\ \mathbf{f}_2 \end{bmatrix} \tag{6.32}$$

Note that the coordinates have been rearranged so that those having the least significant displacements associated with them appear last in the partitioned displacement vector $\mathbf{q}^T = \begin{bmatrix} \mathbf{q}_1^T & \mathbf{q}_2^T \end{bmatrix}$.

Next consider the potential energy of the system defined by the scalar $V_e = (1/2)\mathbf{q}^T K \mathbf{q}$ or, in partitioned form,

$$V_e = \frac{1}{2}\begin{bmatrix} \mathbf{q}_1 \\ \mathbf{q}_2 \end{bmatrix}^T \begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix}\begin{bmatrix} \mathbf{q}_1 \\ \mathbf{q}_2 \end{bmatrix} \tag{6.33}$$

Likewise, the kinetic energy of the system can be written as the scalar $T_e = (1/2)\dot{\mathbf{q}}^T M \dot{\mathbf{q}}$, which becomes

$$T_e = \frac{1}{2}\begin{bmatrix} \dot{\mathbf{q}}_1 \\ \dot{\mathbf{q}}_2 \end{bmatrix}^T \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix}\begin{bmatrix} \dot{\mathbf{q}}_1 \\ \dot{\mathbf{q}}_2 \end{bmatrix} \tag{6.34}$$

in partitioned form. Since each coordinate $\mathbf{q}_i$ is acted upon by a force $\mathbf{f}_i$, the condition that there is no force in the direction of the insignificant coordinates, $\mathbf{q}_2$, requires that $\mathbf{f}_2 = \mathbf{0}$ and that $\partial V_e / \partial \mathbf{q}_2 = 0$. This yields

$$\frac{\partial}{\partial \mathbf{q}_2}(\mathbf{q}_1^T K_{11}\mathbf{q}_1 + \mathbf{q}_1^T K_{12}\mathbf{q}_2 + \mathbf{q}_2^T K_{21}\mathbf{q}_1 + \mathbf{q}_2^T K_{22}\mathbf{q}_2) = 0 \tag{6.35}$$

Solving Equation (6.35) yields a constraint relation between $\mathbf{q}_1$ and $\mathbf{q}_2$ which (since $K_{12} = K_{21}^T$) is as follows:

$$\mathbf{q}_2 = -K_{22}^{-1}K_{21}\mathbf{q}_1 \tag{6.36}$$

This last expression suggests a coordinate transformation (which is *not* a similarity transformation) from the full coordinate system $\mathbf{q}$ to the reduced coordinate system $\mathbf{q}_1$. If the transformation matrix P is defined by

$$P = \begin{bmatrix} I \\ -K_{22}^{-1} \quad K_{21} \end{bmatrix} \tag{6.37}$$

then, if $\mathbf{q} = P\mathbf{q}_1$ is substituted into Equation (6.32) and this expression is premultiplied by P^T, a new reduced-order system of the form

$$P^T MP\ddot{\mathbf{q}}_1 + P^T KP\mathbf{q}_1 = P^T \mathbf{f}_1 \tag{6.38}$$

results. The vector $P^{\mathrm{T}}\mathbf{f}_1$ now has the dimension of $\mathbf{q}_1$. Equation (6.38) represents the reduced-order form of Equation (6.32), where

$$P^T MP = M_{11} - K_{21}^T K_{22}^{-1} M_{21} - M_{12} K_{22}^{-1} K_{21} + K_{21}^T K_{22}^{-1} M_{22} K_{22}^{-1} K_{21} \tag{6.39}$$

and

$$P^T KP = K_{11} - K_{12} K_{22}^{-1} K_{21} \tag{6.40}$$

These last expressions are commonly used to reduce the order of vibration problems in a consistent manner in the case where some of the coordinates (represented by $\mathbf{q}_2$) are thought to be inactive in the system response. This can greatly simplify design and analysis problems in some cases.

If some of the masses in the system are negligible or zero, then the preceding formulae can be used to reduce the order of the vibration problem by setting $M_{22} = 0$ in Equation (6.39). This is essentially the method referred to as *mass condensation* (used in finite element analysis).

Example 6.8.1

Consider a four-degree-of-freedom system with the mass matrix

$$M = \frac{1}{420}\begin{bmatrix} 312 & 54 & 0 & -13 \\ 54 & 156 & 12 & -22 \\ 0 & 13 & 8 & -3 \\ -13 & -22 & -3 & 4 \end{bmatrix}$$

and the stiffness matrix

$$K = \begin{bmatrix} 24 & -12 & 0 & 6 \\ -12 & 12 & -6 & -6 \\ 0 & -6 & 2 & 4 \\ 6 & -6 & 4 & 4 \end{bmatrix}$$

Remove the effect of the last two coordinates. The submatrices of Equation (6.32) are easily identified:

$$M_{11} = \frac{1}{420}\begin{bmatrix} 312 & 54 \\ 54 & 156 \end{bmatrix}, \qquad M_{12} = \frac{1}{420}\begin{bmatrix} 0 & -13 \\ 13 & -22 \end{bmatrix} = M_{21}^T$$

$$M_{22} = \frac{1}{420}\begin{bmatrix} 8 & -3 \\ -3 & 4 \end{bmatrix}, \qquad K_{22} = \begin{bmatrix} 2 & 4 \\ 4 & 4 \end{bmatrix}$$

$$K_{11} = \begin{bmatrix} 24 & -12 \\ -12 & 12 \end{bmatrix}, \qquad K_{12} = \begin{bmatrix} 0 & 6 \\ -6 & -6 \end{bmatrix} = K_{21}^T$$

Using Equations (6.39) and (6.40) yields

$$P^{\mathrm{T}}MP = \begin{bmatrix} 1.021 & 0.198 \\ 0.198 & 0.236 \end{bmatrix}$$
$$P^{\mathrm{T}}KP = \begin{bmatrix} 9 & 3 \\ 3 & 3 \end{bmatrix}$$

These last two matrices form the resulting reduced-order model of the structure.

It is interesting to compare the eigenvalues (frequencies squared) of the full-order system with those of the reduced-order system, remembering that the transformation P used to perform the reduction is *not* a similarity transformation and subsequently does not preserve eigenvalues. The eigenvalues of the reduced system and full-order systems are

$$\lambda_1^{\mathrm{rom}} = 6.981, \qquad \lambda_2^{\mathrm{rom}} = 12.916$$
$$\lambda_1 = 6.965, \qquad \lambda_2 = 12.196$$
$$\lambda_3 = 230.934, \qquad \lambda_4 = 3.833 \times 10^3$$

where the superscript 'rom' refers to the eigenvalues of the reduced-order model. Note that in this case the reduced-order model captures the nature of the first two eigenvalues very well. This is not always the case because the matrix P defined in Guyan reduction, unlike the matrix P from modal analysis, does not preserve the system eigenvalues. More sophisticated model reduction algorithms exist, and some are presented in Section 7.7.

CHAPTER NOTES

A vast amount of literature is available on methods of vibration isolation and absorption. In particular, the books by Balandin, Bolotnik, and Pilkey (2001), Rivin (2003), and by Korenev and Reznikov (1993) should be consulted to augment the information of Section 6.2. The absorber optimization problem discussed in Section 6.3 is directly from the paper of Soom and Lee (1983). Haug and Arora (1976) provide an excellent account of optimal design methods. Example 6.4.1 is from Inman and Jiang (1987). More on the use of damping materials can be found in the book by Nashif, Jones, and Henderson (1985). The material of Section 6.5 comes from Whitesell (1980), which was motivated by the work of Fox and Kapoor (1968). More advanced approaches to eigensystem derivatives can be found in Adhikari and Friswell (2001). The pole placement approach to control can be found in almost any text on control, such as Kuo and Golnaraghi (2003), and is considered in more detail in Section 7.3. The section on design specification (Section 6.7), is an attempt to quantify some of the terminology often used by control and structure researchers in discussing the response of a system. An excellent treatment of reduction of order is given by Meirovitch (1980) and by Antoulas (2005) who provides a mathematical approach. A more advanced treatment of model reduction is given in Section 7.7 from the controls perspective. An excellent summary of model reduction methods, including damped systems, is given by Qu (2004), which contains an extensive bibliography of model reduction papers.

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PROBLEMS

6.1 Calculate the value of the damping ratio required in a vibration isolation design to yield a transmissibility ratio of 0.1 given that the frequency ratio ω/ω_n is fixed at 6.

6.2 A single-degree-of-freedom system has a mass of 200 kg and is connected to its base by a simple spring. The system is being disturbed harmonically at 2 rad/s. Choose the spring stiffness so that the transmissibility ratio is less than 1.

6.3 A spring–mass system consisting of a 10 kg mass supported by a 2000 N m spring is driven harmonically by a force of 20 N at 4 rad/s. Design a vibration absorber for this system and compute the response of the absorber mass.

6.4 Find the minimum and maximum points of the function

$$J(\mathbf{y}) = y_1^3 + 3y_1y_2^2 - 3y_1^2 - 3y_2^2 + 4$$

Which points are actually minimum?

6.5 Calculate the minimum of the cost function

$$J(\mathbf{y}) = y_1 + 2y_2^2 + y_3^2 + y_4^2$$

subject to the equality constraints

$$h_1(\mathbf{y}) = y_1 + 3y_2 - y_3 + y_4 - 2 = 0$$
$$h_2(\mathbf{y}) = 2y_1 + y_2 - y_3 + 2y_4 - 2 = 0$$

6.6 Derive Equations (6.17) and (6.18).

6.7 Derive Equation (6.25). (Hint: first multiply Equation (6.24) by M, then differentiate.)

6.8 Consider Example 6.5.1. Calculate the change in the eigenvalues of this system if the mass, m_1, is changed an unknown amount rather than the stiffness (refer to example 3.3.2).

6.9 Consider example 2.4.4 with $M = I$, $c_1 = 2$, $c_2 = 1$, $k_1 = 4$, and $k_2 = 1$. Calculate a control law causing the closed-loop system to have eigenvalues $\lambda_{1,2} = -1 \pm j$ and $\lambda_{3,4} = -2 \pm j$, using the approach of Section 6.6.

6.10 By using the results of Section 3.6, show that the damping ratio matrix Z' determines whether the modes of a nonproportionally damped system are underdamped or critically damped.

6.11 Consider the system of example 6.4.1 with the damping matrix D set to zero. Calculate a new damping matrix D such that the new system has modal damping ratios $\zeta_1 = 0.1$ and $\zeta_2 = 0.01$.

6.12 Consider the cost function $J(\mathbf{y})$. The partial derivative J with respect to the elements of the vector $\mathbf{y}$ yield only necessary conditions for a minimum. The second-order condition and sufficient condition is that the matrix of second partial derivatives $[J_{ik}]$ be positive definite. Here, J_{ij} denotes the second partial derivative with respect to y_i and y_k. Apply this second condition to problem 6.2 and verify this result for that particular example.

6.13 Derive second-order conditions (see problem 6.10) for example 6.3.1 using a symbolic manipulation program.

6.14 Show that the reduction transformation P of Section 6.8 is not a similarity transformation. Are eigenvalues invariant under P?

6.15 Show that $K_{21} = K_{12}^T$ and hence derive Equation (6.36). In addition, show that $P^T MP$ and $P^T KP$ are both symmetric.

6.16 Calculate a reduced-order model of the following system by removing the last two coordinates:

$$\begin{bmatrix} 312 & 0 & 54 & -6.5 \\ 0 & 2 & 6.5 & -0.75 \\ 54 & 6.5 & 156 & -11 \\ -6.5 & -0.75 & -11 & 1 \end{bmatrix} \ddot{\mathbf{q}}(t) + \begin{bmatrix} 24 & 0 & -12 & 3 \\ 0 & 2 & -3 & \frac{1}{2} \\ -12 & -3 & 12 & -3 \\ 3 & \frac{1}{2} & -3 & 1 \end{bmatrix} \mathbf{q}(t) = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} f(t)$$

Then, calculate the natural frequencies of both the reduced-order and the full-order system (using a code such as MATLAB) and compare them. Also, plot the response of each system to the initial conditions $\mathbf{q} = [1 \quad 0 \quad 0 \quad 0]^T$ and $\dot{\mathbf{q}} = \mathbf{0}$ and compare the results.

6.17 The characteristic equation of a given system is

$$\lambda^3 + 5\lambda^2 + 6\lambda + \eta = 0$$

where η is a design parameter. Calculate the stability margin of this system for $\eta_{op} = 15.1$.

6.18 Compare the time response of the coordinates $q_1(t)$ and $q_2(t)$ of the full-order system of example 6.8.1 with the same coordinates in the reduced-order system for an initial displacement of $q_1(0) = 1$ and all other initial conditions set to zero.

6.19 Consider the system of example 6.4.1 with the damping matrix set to zero. Use the pole placement approach of Section 6.6 to compute a control law that will cause the closed-loop system to have frequencies of 2 and 3 rad/s.

6.20 Consider the vibration absorber designed in problem 6.3. Use numerical simulation to plot the response of the system to an initial 0.01 m displacement disturbance of m_1 (zero initial velocity). Discuss your results.