9 Distributed-parameter Models

9.1 INTRODUCTION

This chapter presents an informal introduction to the vibrations of systems having distributed mass and stiffness, often called distributed-parameter systems. For lumped-parameter systems, the single-degree-of-freedom system served as a familiar building block with which more complicated multiple-degree-of-freedom structures can be modeled. Similarly, an examination of the vibrations of simple models of strings, beams, and plates provides a set of ‘building blocks’ for understanding the vibrations of systems with distributed mass, stiffness, and damping parameters. Such systems are referred to as distributed-parameter systems, elastic systems, continuous systems, or flexible systems.

This chapter focuses on the basic methods used to solve the vibration problem of flexible systems. The purpose of this chapter is to list the equations governing the linear vibrations of several distributed-parameter structures, list the assumptions under which they are valid, and discuss some of the possible boundary conditions for such structures. The equations are not derived; they are simply stated with a few representative examples of solutions. References such as Meirovitch (2001) and Magrab (1979) should be consulted for derivations of the equations of motion. These solution methods are made rigorous and discussed in more detail in Chapter 10.

9.2 VIBRATION OF STRINGS

Figure 9.1 depicts a string fixed at both ends and displaced slightly from its equilibrium position. The lateral position of the string is denoted by w. The value of w will depend not only on the time t but also on the position along the string x. This spatial dependency is the essential difference between lumped-parameter systems and distributed-parameter systems – namely the deflection of the string is a function of both x and t and hence is denoted by $w = w(x, t)$.

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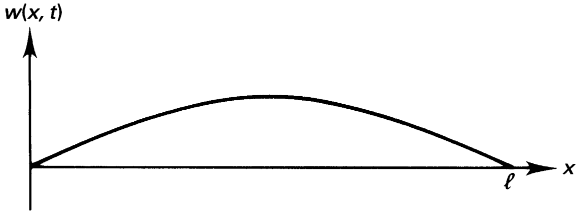

Figure 9.1 String fixed at both ends and displaced from its equilibrium position.

For small deflections and slopes of the string where the restoring forces in the vertical displacement of the string are due entirely to the axial tension T in the string, the equation governing the function $w(x, t)$ can be shown to be

$$w_{xx}(x, t) - \frac{1}{c^2} w_{tt}(x, t) \tag{9.1}$$

where $c = \sqrt{T/\rho}$, ρ is the mass per unit length of the string, w_{xx} denotes the second partial derivative of $w(x, t)$ with respect to x, and w_{tt} denotes the second partial derivative of $w(x, t)$ with respect to t. Alternatively, these partial derivatives are denoted by $\partial^2/\partial x^2$ and $\partial^2/\partial t^2$. The derivation of this equation can be found in many texts (see, for instance, Inman, 2001). Briefly, Equation (9.1) comes from applying a simple force balance on an infinitesimal element of the string. The quantity w_{xx} is the derivative of the slope and hence is proportional to a restoring force. The right-hand side of Equation (9.1) is just the acceleration multiplied by a coefficient and hence the inertia of the string.

The function $w(x, t)$ that satisfies Equation (9.1) must also satisfy two initial conditions because of the second-order time derivatives. The second-order spatial derivative implies that two spatial conditions must also be satisfied. In the cases of interest here, the value of x will vary over a finite range. Physically, if the string is fixed at both ends, then $w(x, t)$ would have to be zero at $x = 0$ and again at $x = \ell$. These conditions are known as boundary conditions. Mathematically, there must be one boundary condition (or one constant of integration) for each spatial derivative and one initial condition for each time derivative.

The problem of finding the lateral vibrations of a string fixed at both ends can then be summarized as follows. Find a function $w(x, t)$ such that

$$\begin{aligned}
&w_{xx}(x, t) = \frac{1}{c^2} w_{tt}(x, t), \qquad x \in (0, \ell) \qquad \text{for } t > 0 \\
&w(0, t) = w(\ell, t) = 0, \qquad t > 0 \\
&w(x, 0) = w_0(x) \qquad \text{and} \qquad w_t(x, 0) = \dot{w}_0(x) \qquad \text{at } t = 0
\end{aligned} \tag{9.2}$$

where $w_0(x)$ and $\dot{w}_0(x)$ are specified time-invariant functions representing the initial (at $t = 0$) displacement and velocity distribution of the string. The notation $x \in (0, \ell)$ means that the equation holds for values of x in the interval $(0, \ell)$.

One approach used to solve the system given by Equations (9.2) is to assume that the solution has the form $w(x, t) = X(x)T(t)$. This approach is called the method of *separation of variables* (see, for instance, Boyce and DiPrima, 2000). The method proceeds by substitution of this assumed separated form into Equation (9.2), which yields the set of equations

$$
\begin{aligned}
&X''(x)T(t) = \frac{1}{c^2}X(x)\ddot{T}(t) \\
&X(0)T(t) = 0 \qquad \text{and} \qquad X(\ell)T(t) = 0 \\
&X(x)T(0) = w_0(x) \qquad \text{and} \qquad X(x)\dot{T}(0) = \dot{w}_0(x)
\end{aligned} \tag{9.3}
$$

where the primes denote total derivatives with respect to x and the overdots indicate total time differentiation. Rearranging the first equation in system (9.3) yields

$$\frac{X''(x)}{X(x)} = \frac{1}{c^2}\frac{\ddot{T}(t)}{T(t)} \tag{9.4}$$

Differentiating this expression with respect to x yields

$$\frac{d}{dx}\left(\frac{X''(x)}{X(x)}\right) = 0$$

or that

$$\frac{X''(x)}{X(x)} = \sigma \tag{9.5}$$

where σ is a constant (independent of t or x) of integration. The next step then is to solve Equation (9.5), i.e.,

$$X''(x) - \sigma X(x) = 0 \tag{9.6}$$

subject to the two boundary conditions, which become

$$X(0) = 0 \qquad \text{and} \qquad X(\ell) = 0 \tag{9.7}$$

since $T(t) \neq 0$ for most values of t.

The nature of the constant σ needs to be determined next. There are three possible choices for σ: it can be positive, negative, or zero. If $\sigma = 0$, the solution of Equation (9.6) subject to Equation (9.7) becomes $X(x) = 0$, which does not satisfy the condition of a nontrivial solution. If $\sigma > 0$, then the solution to Equation (9.6) is of the form $X(x) = A_1 \cosh(\sigma x) + A_2 \sinh(\sigma x)$. Applying the initial conditions to this form of the solution then yields

$$
\begin{aligned}
0 &= A_1 \\
0 &= A_2 \sinh(\ell\sigma)
\end{aligned}
$$

so that again the only possible solution is the trivial solution, $w(x, t) = 0$.

Thus, the only nontrivial choice for σ is that it must have a negative value. To indicate this, $\sigma = -\lambda^2$ is used. Equations (9.6) and (9.7) become

$$X''(x) + \lambda^2 X(x) = 0$$
$$X(0) = X(\ell) = 0 \tag{9.8}$$

Equations (9.8) are called a *boundary value problem* as the values of the solution are specified at the boundaries. The form of the solution of Equations (9.8) is

$$X(x) = A_1 \sin \lambda x + A_2 \cos \lambda x \tag{9.9}$$

Applying the boundary conditions to the solution (9.9) indicates that the constants A_1 and A_2 must satisfy

$$A_2 = 0 \qquad \text{and} \qquad A_1 \sin \lambda\ell = 0 \tag{9.10}$$

Examination of Equation (9.10) shows that the only nontrivial solutions occur if

$$\sin \lambda\ell = 0 \tag{9.11}$$

or when λ has the value $n\pi/\ell$, where n is any integer value, denoted by

$$\lambda_n = \frac{n\pi}{\ell}, \qquad n = 1, 2, \ldots, \infty$$

Note that the $n = 0$, $\lambda = 0$ is omitted in this case because it results in zero solution. The solution given in Equation (9.9) is thus the infinite set of functions denoted by

$$X_n(x) = A_n \sin\left(\frac{n\pi}{\ell} x\right) \tag{9.12}$$

where n is any positive integer. Equation (9.11) resulting from applying the boundary conditions is called the *characteristic equation* and the values λ_n are called *characteristic values*.

Substitution of Equation (9.12) into Equation (9.4) then shows that the temporal coefficient $T(t)$ must satisfy the infinite number of equations

$$\ddot{T}(t) + \left(\frac{n\pi}{\ell}\right)^2 c^2 T(t) = 0 \tag{9.13}$$

The solution of these equations, one for each n, is given by

$$T_n(t) = A_n^1 \sin\left(\frac{n\pi c}{\ell} t\right) + A_n^2 \cos\left(\frac{n\pi c}{\ell} t\right) \tag{9.14}$$

where A_n^1 and A_n^2 are the required constants of integration and the subscript n has been added to indicate that there are an infinite number of solutions of this form. Thus, the solutions of Equation (9.1), also infinite in number, are of the form

$$w_n(x, t) = a_n \sin\left(\frac{n\pi c}{\ell} t\right) \sin\left(\frac{n\pi}{\ell} x\right) + b_n \cos\left(\frac{n\pi c}{\ell} t\right) \sin\left(\frac{n\pi}{\ell} x\right) \tag{9.15}$$

Here, a_n and b_n are constants representing the product of A_n^1 and A_n^2 of Equation (9.14) and the constants of Equation (9.12).

Since Equation (9.15) is a linear system, the sum of all of these solutions is also a solution, so that

$$w(x,t)=\sum_{n=1}^{\infty}\left[a_n\sin\left(\frac{n\pi c}{\ell}t\right)+b_n\cos\left(\frac{n\pi c}{\ell}t\right)\right]\sin\left(\frac{n\pi}{\ell}x\right) \tag{9.16}$$

This infinite sum may or may not converge, as is discussed in Chapter 11. Next, the constants a_n and b_n need to be calculated. These constants come from the initial conditions. Applying the displacement initial condition to Equation (9.16) yields

$$w(x,0)=w_0(x)=\sum_{n=1}^{\infty}b_n\sin\left(\frac{n\pi}{\ell}x\right) \tag{9.17}$$

This equation can be solved for the constants b_n by using the (orthogonality) property of the functions $\sin(n\pi/\ell)x$:

$$\int_0^{\ell}\sin\left(\frac{n\pi}{\ell}x\right)\sin\left(\frac{m\pi}{\ell}x\right)dx=\delta_{mn}=\begin{cases}\dfrac{1}{2}, & m=n\\ 0, & m\neq n\end{cases} \tag{9.18}$$

Thus, multiplying Equation (9.17) by $\sin(m\pi x/\ell)$ and integrating yields the desired constants

$$b_n=\frac{2}{\ell}\int_0^{\ell}w_0(x)\sin\left(\frac{n\pi}{\ell}x\right)dx \tag{9.19}$$

since the sum vanishes for each $n\neq m$. Likewise, if Equation (9.16) is differentiated with respect to time, multiplied by $\sin(m\pi x/\ell)$, and integrated, the constants a_n are found from

$$a_n=\frac{2}{n\pi c}\int_0^{\ell}\dot{w}_0(x)\sin\left(\frac{n\pi}{\ell}x\right)dx \tag{9.20}$$

The solution to problem (9.2) is given by the relations (9.16), (9.19), and (9.20).

The problem of solving the most basic distributed-parameter system is much more complicated than solving for the free response of a simple one-degree-of-freedom lumped-parameter system. Also, note that the solution just described essentially yields the theoretical modal analysis solution established for lumped-parameter systems as developed in Section 3.3. The functions $\sin(n\pi x/\ell)$ serve in the same capacity as the eigenvectors of a matrix in calculating a solution. The major difference between the two developments is that the sum in Equation (3.42) is finite, and hence always converges, whereas the sum in Equation (9.16) is infinite and may or may not converge.

Physically, the functions $\sin(n\pi x/\ell)$ describe the configuration of the string for a fixed time and hence are referred to as the *natural modes of vibration* of the system. Likewise, the numbers $(n\pi c/\ell)$ are referred to as the *natural frequencies of vibration*, since they describe the motion periodicity in time. Mathematically, the characteristic values $(n\pi/\ell)$ are also called the *eigenvalues* of the system, and sin $(n\pi x/\ell)$ are called the *eigenfunctions* of the system and form an analogy to what is known about lumped-parameter systems. These quantities are defined more precisely in Section 10.2. For now, note that the eigenvalues and eigenfunctions defined here serve the same purpose as eigenvalues and eigenvectors defined for matrices.

The basic method of separation of variables combined with the infinite sums and orthogonality as used here forms the basic approach in solving for the response of distributed-parameter systems. This approach, also called modal analysis, is used numerous times in the following chapters to solve a variety of vibration and control problems.

In the case of lumped-parameter systems, a lot was gained by looking at the eigenvalue and eigenvector problem. This information allowed the calculation of the solution of both the free and forced response by using the properties of the eigenstructures. In the following, the same approach (modal analysis) is further developed for distributed-parameter systems.

The fact that the solution (9.16) is a series of sine functions should not be a surprise. In fact, Fourier's theorem states that every function $f(x)$ that is piecewise continuous and bounded on the interval $[a, b]$ can be represented as a Fourier series of Equation (8.1), i.e.,

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos\left(\frac{n\pi}{\ell}x\right) + b_n \sin\left(\frac{n\pi}{\ell}x\right) \right] \tag{9.21}$$

where $\ell = b - a$. This fact is used extensively in Chapter 8 on vibration testing.

Recall that a function f is said to be *bounded* on the interval $[a, b]$ if there exists a finite constant M such that $|f(x)| < M$ for all x in $[a, b]$. Furthermore, a function $f(x)$ is *continuous* on the interval $[a, b]$ if for every x_1 in $[a, b]$, and for every $\varepsilon > 0$, there exists a number $\delta = \delta(\varepsilon) > 0$ such that $|x - x_1| < \delta$ implies $|f(x) - f(x_1)| < \varepsilon$. A function is *piecewise continuous* on $[a, b]$ if it is continuous on every subinterval of $[a, b]$ (note here that the square brackets indicate that the endpoints of the interval are included in the interval).

Note that in many cases either all of the coefficients a_n or all of the coefficients b_n are zero. Also, note that many other functions $\Theta_n(x)$ besides the functions $\sin(n\pi x/\ell)$ and $\cos(n\pi x/\ell)$ have the property that arbitrary functions of a certain class can be expanded in terms of an infinite series of such functions, i.e., that

$$f(x) = \sum_{n=1}^{\infty} a_n \Theta_n(x) \tag{9.22}$$

This property is called *completeness* and is related to the idea of completeness used with orthogonal eigenvectors (Section 3.3). This concept is discussed in detail in Chapter 10.

Note that Equation (9.22) really means that the sequence of partial sums

$$\left\{ a_1\Theta_1, a_1\Theta_1 + a_2\Theta_2, \ldots, \sum_{i=1}^{m} a_i\Theta_i, \ldots \right\} \tag{9.23}$$

converges to the function $f(x)$, i.e., that

$$\lim_{m\to\infty} \left(\sum_{n=1}^{m} a_n\Theta_n \right) = f(x) \tag{9.24}$$

as defined in most introductory calculus texts.

Example 9.2.1

Now that the formal solution of the string has been examined and the eigenvalues and eigenfunctions have been identified, it is important to realize that these quantities are the physical notions of mode shapes and natural frequencies. To this end, suppose that the string is given the following initial conditions:

$$w(x,0)\sin\left(\frac{\pi}{\ell}x\right), \qquad w_t(x,0)=0 \tag{9.25}$$

Calculation of the expansion coefficients yields [from Equations (9.19) and (9.20)]

$$a_n=\frac{2}{n\pi c}\int_0^\ell w_t(x,0)\sin\left(\frac{n\pi}{\ell}x\right)dx=0 \tag{9.26}$$

and

$$b_n=\frac{2}{\ell}\int_0^\ell \sin\left(\frac{\pi}{\ell}x\right)\sin\left(\frac{n\pi}{\ell}x\right)dx=\begin{cases}1, & n=1\\ 0, & n>2\end{cases} \tag{9.27}$$

The solution thus becomes

$$w(x,t)=\sin\left(\frac{\pi x}{\ell}\right)\cos\left(\frac{\pi ct}{\ell}\right) \tag{9.28}$$

In Figure 9.2, this solution is plotted versus x for a fixed value of t. This plot is the *shape* that the string would take if it were viewed by a stroboscope blinking at a frequency of $\pi c/\ell$. One of the two curves in Figure 9.2 is the plot of $w(x,t)$ that would result if the string were given an initial displacement of $w(x,0)=\sin(\pi x/\ell)$, the first eigenfunctions. The other curve, which takes on negative values, results from an initial condition of $w(x,0)=\sin(2\pi x/\ell)$, the second eigenfunction. Note that all the eigenfunctions $\sin(n\pi x/\ell)$ can be generated in this fashion by choosing the appropriate initial conditions. Hence, this set of functions is known as the set of *mode shapes* of vibration of the string. These correspond to the mode shapes defined for lumped-parameter systems and are the quantities measured in the modal tests described in Chapter 8.

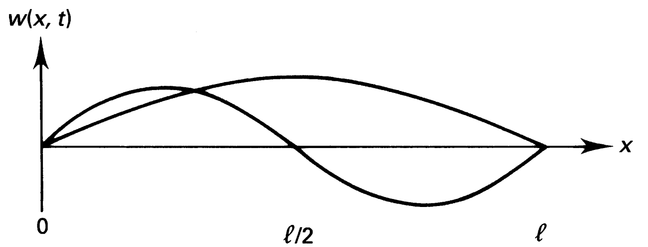

Figure 9.2 First two mode shapes of a vibrating string fixed at both ends.

9.3 RODS AND BARS

Next, consider the longitudinal vibration of a bar – that is, the vibration of a long slender material in the direction of its longest axis, as indicated in Figure 9.3. Again, by summing forces, the equation of motion is found (see, for instance, Timoshenko, Young, and Weaver, 1974) to be

$$[EA(x)w_x(x,t)]_x = \rho(x)A(x)w_{tt}(x,t), \qquad x \in (0,\ell) \tag{9.29}$$

where $A(x)$ is the variable cross-sectional area, $\rho(x)$ represents the variable mass distribution per unit area, E is the elastic modulus, and $w(x,t)$ is the axial displacement (in the x direction).

The form of Equation (9.29) is the same as that of the string if the cross-sectional area of the bar is constant. In fact, the 'stiffness' *operator* (see Section 10.2) in both cases has the form

$$-\alpha \frac{\partial^2}{\partial x^2} \tag{9.30}$$

where α is a constant. Hence, the eigenvalues and eigenfunctions are expected to have the same mathematical form as those of the string, and the solution will be similar. The main difference between these two systems is physical. In Equation (9.29) the function $w(x,t)$ denotes displacements along the long *axis* of the rod, where, as in Equation (9.1), $w(x,t)$ denotes displacements *perpendicular to the axis* of the string.

Several different ways of supporting a rod (and a string) lead to several different sets of boundary conditions associated with Equation (9.29). Some are stated in terms of the displacement $w(x,t)$ and others are given in terms of the strain $w_x(x,t)$ (recall that strain is the change in length per unit length).

Free boundary. If the bar is free or unsupported at a boundary, then the stress at the boundary must be zero, i.e., no force should be present at that boundary, or

$$EA(x)\left.\frac{\partial w(x,t)}{\partial x}\right|_{x=0} = 0 \qquad \text{or} \qquad EA(x)\left.\frac{\partial w(x,t)}{\partial x}\right|_{x=\ell} = 0 \tag{9.31}$$

Note that, if $A(\ell) \neq 0$, the strain $w_x(\ell,t)$ must also be zero. The vertical bar in Equation (9.31) denotes that the function is to be evaluated at $x=0$ or ℓ after the derivative is taken and indicates the location of the boundary condition.

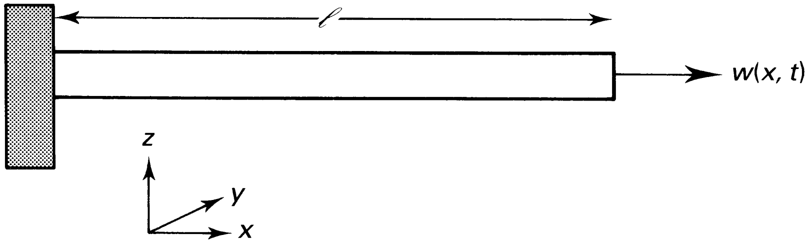

Figure 9.3 Schematic of a rod or bar, indicating the direction of longitudinal vibration.

Clamped boundary. If the boundary is rigidly fixed, or clamped, then the displacement must be zero at that point or

$$w(x, t)|_{x=0} = 0 \qquad \text{or} \qquad w(x, t)\,|_{x=\ell} = 0 \tag{9.32}$$

Appended boundary. If the boundary is fastened to a lumped element, such as a spring of stiffness k, the boundary condition becomes

$$EA(x)\frac{\partial w(x, t)}{\partial x}|_{x=0} = -kw(x, t)|_{x=0} \qquad \text{or} \qquad EA(x)\frac{\partial w(x, t)}{\partial x}|_{x=\ell} = kw(x, t)|_{x=\ell} \tag{9.33}$$

which expresses a force balance at the boundary. In addition, if the bar has a lumped mass at the end, the boundary condition becomes

$$EA(x)w_x(x, t)\,|_{x=0} = mw_{tt}(x, t)\,|_{x=0} \qquad \text{or} \qquad EA(x)w_x(x, t)\,|_{x=\ell} = -mw_{tt}(x, t)\,|_{x=\ell} \tag{9.34}$$

which also represents a force balance. These types of problems are discussed in detail in Section 12.4. They represent a large class of applications and are also referred to as structures with time-dependent boundary conditions, constrained structures, or combined dynamical systems.

As noted, the equation of motion is mathematically the same for both the string and the bar, so that further discussion of the method of solution is not required. A third problem again has the same mathematical model: the torsional vibration of circular shafts (rods). The derivation of the equation of motion is very similar, comes from a force balance, and can be found in several references (see, for instance, Timoshenko, Young, and Weaver, 1974). If G represents the shear modulus of elasticity of the shaft, I_p is the polar moment of inertia of the shaft, ρ is the mass per unit area, $\theta(x, t)$ is the angular displacement of the shaft from its neutral position, and x is the distance measured along the shaft, then the equation governing the torsional vibration of the shaft is

$$\theta_{tt}(x, t) = \frac{G}{\rho}\theta_{xx}(x, t), \qquad x \in (0, \ell) \tag{9.35}$$

As in the case of the rod or bar, the shaft can be subjected to a variety of boundary conditions, some of which are described in the following.

Free boundary. If the boundaries of the shaft are not attached to any device, there cannot be any torque acting on the shaft at that point, so that

$$GI_p\theta_x\,|_{x=0} = 0 \qquad \text{or} \qquad GI_p\theta_x\,|_{x=\ell} = 0 \tag{9.36}$$

at that boundary. If G and I_p are constant, then Equation (9.36) becomes simply

$$\theta_x(x, t)\,|_{x=0} = 0 \qquad \text{or} \qquad \theta_x(x, t)\,|_{x=\ell} = 0 \tag{9.37}$$

Clamped boundary. If a boundary is clamped, then no movement of the shaft at that position can occur, so that the boundary condition becomes

$$\theta(x, t)\,|_{x=0} = 0 \qquad \text{or} \qquad \theta(x, t)\,|_{x=\ell} = 0 \tag{9.38}$$

Appended boundaries. If a torsional spring of stiffness k is attached at the right end of the shaft (say at $x = \ell$), then the spring force (torque) must balance the internal bending moment. The boundary condition becomes

$$GI_p\theta(x, t)\,|_{x=\ell} = k\theta(x, t)\,|_{x=\ell} \tag{9.39}$$

At the left end this becomes (see Figure 9.4)

$$GI_p\theta_x(x, t)\,|_{x=0} = -k\theta(x, t)\,|_{x=0} \tag{9.40}$$

Quite often a shaft is connected to a disc at one end or the other. If the disc has mass polar moment of inertia I_d at the right end, the boundary condition becomes (see Figure 9.4)

$$GI_p\theta_x(x, t)\,|_{x=\ell} = -I_d\theta_{tt}(x, t)\,|_{x=\ell} \tag{9.41}$$

or

$$GI_p\theta_x(x, t)\,|_{x=0} = I_d\theta_{tt}(x, t)\,|_{x=0} \tag{9.42}$$

if the disc is placed at the left end.

The shaft could also have both a spring and a mass at one end, in which case the boundary condition, obtained by summing forces, is

$$GI_p\theta_x(x, t)\,|_{x=0} = -I_d\theta_{tt}(x, t)\,|_{x=0} - k\theta(x, t)\,|_{x=0} \tag{9.43}$$

As illustrated in the examples and problems, the boundary conditions affect the natural frequencies and mode shapes. These quantities have been tabulated for many common boundary conditions (Gorman, 1975, and Blevins, 2001).

Example 9.3.1

Consider the vibration of a shaft that is fixed at the left end ($x = 0$) and has a disc attached to the right end ($x = \ell$). Let G, I_p, and ρ all have unit values, and calculate the eigenvalue and eigenfunctions of the system.

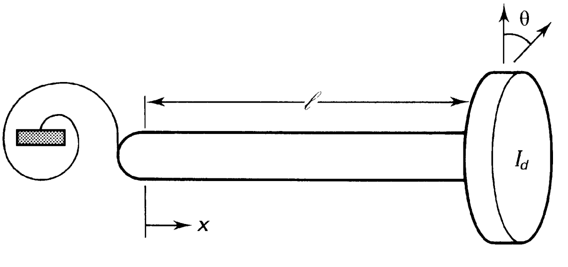

Figure 9.4 Shaft connected to a rotational spring on one end and a disc on the other end.

Following the separation of variables procedure used in the solution of the string problems, a solution of the form $\theta(x, t) = \Theta(x)T(t)$ is assumed and substituted into the equation of motion and boundary conditions, resulting in

$$\Theta''(x) + \lambda^2 \Theta(x) = 0$$
$$\Theta(0) = 0$$
$$GI_p \Theta'(\ell)T(t) = I_d \Theta(\ell)\ddot{T}(t)$$

Recall that $T(t)$ is harmonic, so that $\ddot{T}(t) = -\lambda^2 T(t)$, and the last boundary condition can be written as

$$GI_p \Theta'(\ell) = -\lambda^2 I_d \Theta(\ell)$$

which removes the time dependence. The general spatial solution is

$$\Theta(x) = A_1 \sin \lambda x + A_2 \cos \lambda x$$

Application of the boundary condition at $x = 0$ yields

$$A_2 = 0 \qquad \text{and} \qquad \Theta(x) = A_1 \sin \lambda x$$

The second boundary condition yields (for the case $G = I_p = 1$)

$$\lambda A_1 \cos \lambda \ell = -\lambda^2 I_d A_1 \sin \lambda \ell$$

which is satisfied for all values of λ such that

$$\tan(\lambda \ell) = -\frac{\ell}{I_d} \frac{1}{\lambda \ell} \tag{9.44}$$

Equation (9.44) is a transcendental equation for the values of λ, the eigenvalues, and has an infinite number of solutions denoted by λ_n, calculated either graphically or numerically from the points of intersection given in Figure 9.5. The values of λ correspond to the values of $(\lambda \ell)$ at the intersections

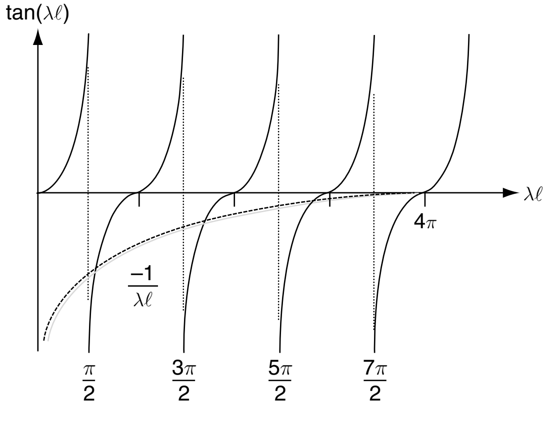

Figure 9.5 Graphical solution of the transcendental equation for tan $\lambda \ell = -1/\lambda \ell$.

of the two curves of Figure 9.5. Note that the effect of the disc inertia, I_d, is to shift the points of intersection of the two curves. The transcendental equation [Equation (9.44)] is solved numerically near each crossing, using the plot to obtain an initial guess for the numerical procedure. For values of n greater than 3, the crossing points in the plot approach the zeros of the tangent, and hence the sine or $n\pi$.

9.4 VIBRATION OF BEAMS

In this section, the transverse vibration of a beam is examined. A beam is represented in Figure 9.6. The beam has mass density $\rho(x)$, cross-sectional area $A(x)$, and moment of inertia $I(x)$ about its equilibrium axis. The deflection $w(x,t)$ of the beam is the result of two effects, bending and shear. The derivation of equations and boundary conditions for the vibration of the beam can be found in several texts (see, for instance, Timoshenko, Young, and Weaver, 1974).

A beam is a transversely loaded (along the z axis in Figure 9.6), prismatic structural element with length ℓ, which is large in value when compared with the magnitude of the beam cross-sectional area, A (the y–z plane in Figure 9.6). In the previous section, vibration of such a structure in the x direction was considered and referred to as the longitudinal vibration of a bar. In this section the *transverse vibration* of the beam, i.e., vibration in the z direction, perpendicular to the long axis of the beam, is considered.

The equations of motion are not developed here; however, the three basic assumptions used in the derivation are important to bear in mind. The fundamental small displacement assumption of linear vibration theory in this case results in requiring (1) that the material in the y–z plane of the beam remains in the plane during deformation, (2) that the displacements along the y direction are zero (called the *plane strain* assumption), and (3), that, along any cross-section, the displacement in the z direction is the same (i.e., no stretch of material in thickness). These assumptions lead to the *Timoshenko beam equations* in the transverse

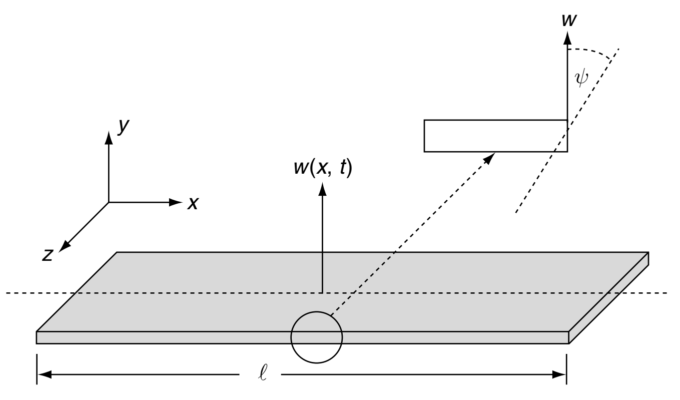

Figure 9.6 Beam indicating the variables used in the transverse vibration model.

deflection $w(x, t)$ and the bending slope $\psi(x, t)$ (see, for example, Reismann and Pawlik, 1974). They are

$$
\begin{aligned}
&-\rho A w_{tt} + \frac{\partial}{\partial x}\left[\kappa^2 AG\left(\frac{\partial w}{\partial x} - \psi\right)\right] + p = 0, \qquad x \in (0, \ell) \\
&-\rho I \psi_{tt} + \frac{\partial}{\partial x}\left[EI\frac{\partial \psi}{\partial x}\right] + \kappa^2 AG\left(\frac{\partial w}{\partial x} - \psi\right) = 0, \qquad x \in (0, \ell)
\end{aligned}
\tag{9.45}
$$

where κ^2 is called the shear coefficient, G is the shear modulus, E is the elastic modulus, and $p = p(x, t)$ is an externally applied force along the length of the beam. The shear coefficient κ^2 can be determined in several ways. A summary of values for κ^2 is given by Cowper (1966).

With the additional assumptions that (1) shear deformations are negligible, so that the shear angle is zero and $\psi = -w_x(x, t)$, and (2) that the rotary inertia is negligible, so that $\rho I \psi_{tt} = 0$, the so-called *Euler–Bernoulli* beam equations result. Under these additional assumptions, Equations (9.45) become the single equation (for $p = 0$ and A constant)

$$
\frac{\partial^2}{\partial x^2}\left[EI(x)\frac{\partial^2 w(x, t)}{\partial x^2}\right] = -m(x) w_{tt}(x, t), \qquad x \in (0, \ell) \tag{9.46}
$$

where $m(x) = \rho A$. Equation (9.46) is the *Euler–Bernoulli beam equation* and is more commonly used in applications than the Timoshenko equations [Equation (9.45)] because of its comparative simplicity. If the beam is long and skinny (aspect ratio, say, greater then 10), the Euler–Bernoulli assumptions are appropriate. The rest of the section is devoted to the Euler–Bernoulli model. Again, many boundary conditions are possible, and several common cases are considered here.

Clamped boundary. If the beam is firmly fixed at a boundary $x = 0$, then both the deflection of the beam and the slope of the deflection must be zero, i.e.,

$$
w(x, t)|_{x=0} = w_x(x, t)|_{x=0} = 0 \tag{9.47}
$$

Simply supported boundary. If the beam is supported by a hinge at $x = 0$, then the bending moment and deflection must both be zero at that point, i.e.,

$$
\begin{aligned}
EI(x)\frac{\partial^2 w(x, t)}{\partial x^2}\Big|_{x=0} &= 0 \\
w(x, t)|_{x=0} &= 0
\end{aligned}
\tag{9.48}
$$

This boundary condition is also referred to as *hinged*.

Free boundary. Again, in the case of no support at a boundary, the bending moment must be zero. In addition, the shearing force at a free end must be zero, i.e.,

$$
EI(x)\left.\frac{\partial^2 w(x, t)}{\partial x^2}\right|_{x=0} = 0, \qquad \left.\frac{\partial}{\partial x}\left[EI(x)\frac{\partial^2 w(x, t)}{\partial x^2}\right]\right|_{x=0} = 0 \tag{9.49}
$$

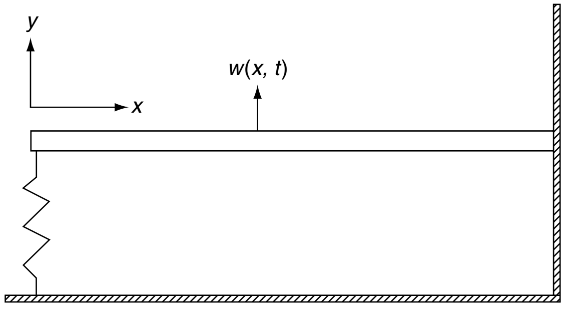

Figure 9.7 Beam connected to a linear spring on one end and clamped on the other end.

Appended boundaries. If a spring, with spring constant k, is attached to one end of the beam as shown in Figure 9.7, a force balance indicates that the shear force must equal the spring-restoring force. In addition, the bending moment must still be zero.

$$EI(x)\left.\frac{\partial^2 w(x,t)}{\partial x^2}\right|_{x=0} = 0, \qquad \left.\frac{\partial}{\partial x}\left[EI(x)\frac{\partial^2 w(x,t)}{\partial x^2}\right]\right|_{x=0} = -kw(x,t)\,|_{x=0} \tag{9.50}$$

Note that the sign again changes if the spring is attached at the other end. If a mass is placed at one end, the shear force must balance the inertial force provided by the mass, and again the bending moment must be zero. Using the same end as indicated in Figure 9.7, the boundary conditions are

$$EI(x)\left.\frac{\partial^2 w(x,t)}{\partial x^2}\right|_{x=0} = 0, \qquad \left.\frac{\partial}{\partial x}\left[EI(x)\frac{\partial^2 w(x,t)}{\partial x^2}\right]\right|_{x=0} = mw_{tt}(x,t)\,|_{x=0} \tag{9.51}$$

where m is the mass of the appended piece.

Example 9.4.1

Consider a cantilevered beam (clamped at $x = 0$ and free at the $x = l$) and compute the natural frequencies and mode shapes using separation of variables as described above. Assume that E, I, ρ, and l are constant. The method is basically the same as that used for a string, rod, and bar, but the resulting boundary value problem is slightly more complex. The clamped–free boundary conditions are

$$w(0,t) = w_x(0,t) = 0 \qquad \text{and} \qquad w_{xx}(l,t) = w_{xxx}(l,t) = 0$$

The equation of motion is

$$\frac{\partial^2 w}{\partial t^2} + \left(\frac{EI}{\rho A}\right)\frac{\partial^4 w}{\partial x^4} = 0$$

Using the method of separation of variables, assume the solution is of the form $w(x,t) = X(x)T(t)$ to obtain

$$\left(\frac{EI}{\rho A}\right)\frac{X''''}{X} = -\frac{\ddot{T}(t)}{T(t)} = \omega^2$$

The spatial equation becomes

$$X''''(x) - \left(\frac{\rho A}{EI}\right)\omega^2 X(x) = 0$$

Next, define $\beta^4 = (\rho A \omega^2)/(EI)$ so that the equation of motion becomes $X''' - \beta^4 X = 0$ which has the solution

$$X(x) = C_1 \sin \beta x + C_2 \cos \beta x + C_3 \sinh \beta x + C_4 \cosh \beta x$$

Applying the boundary conditions in separated form

$$X(0) = X'(0) = 0 \qquad \text{and} \qquad X''(l) = X'''(l) = 0$$

yields four equations in the four unknown coefficients C_i. These four equations are written in matrix form as

$$\begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ -\sin \beta l & -\cos \beta l & \sinh \beta l & \cosh \beta l \\ -\cos \beta l & \sin \beta l & \cosh \beta l & \sinh \beta l \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \end{bmatrix} = 0$$

For a nonzero solution for the coefficients C_i, the matrix determinant must be zero. The determinant yields the characteristic equation

$$(-\sin \beta l - \sinh \beta l)(\sin \beta l - \sinh \beta l) - (-\cos \beta l - \cosh \beta l)(-\cos \beta l - \cosh \beta l) = 0$$

Simplifying, the characteristic equation becomes $\cos \beta l \cosh \beta l = -1$, or

$$\cos \beta_n l = -\frac{1}{\cosh \beta_n l}$$

This last expression is solved numerically for the values βl. The frequencies are then

$$\omega_n = \sqrt{\frac{\beta_n^4 EI}{\rho A}}$$

The mode shapes given by the solution for each β_n are

$$X_n = C_{1n} \sin \beta_n x + C_{2n} \cos \beta_n x + C_{3n} \sinh \beta_n x + C_{4n} \cosh \beta_n x$$

Using the boundary condition information that $C_4 = -C_2$ and $C_3 = -C_1$ yields

$$-C_1(\sin \beta l + \sinh \beta l) = C_2(\cos \beta l + \cosh \beta l)$$

so that

$$C_1 = -C_2\left(\frac{\cos \beta l + \cosh \beta l}{\sin \beta l + \sinh \beta l}\right)$$

The mode shapes can then be expressed as

$$X_n = -C_{2n}\left[-\left(\frac{\cos \beta_n l + \cosh \beta_n l}{\sin \beta_n l + \sinh \beta_n l}\right)\sin \beta_n x + \cos \beta_n x + \left(\frac{\cos \beta_n l + \cosh \beta_n l}{\sin \beta_n l + \sinh \beta_n l}\right)\sinh \beta_n x - \cosh \beta_n x\right]$$

Additional examples of the solution to the beam equation can be found in the next chapter which discusses formal methods of solution.

9.5 MEMBRANES AND PLATES

In this section, the equations for linear vibrations of membranes and plates are discussed. These objects are two-dimensional versions of the strings and beams discussed in the preceding sections. They occupy plane regions in space. The membrane represents a two-dimensional version of a string, and a plate can be thought of as a membrane with bending stiffness.

First, consider the equations of motion for a membrane. A membrane is basically a two-dimensional system that lies in a plane when in equilibrium. A common example is a drum head. The structure itself provides no resistance to bending, so that the restoring force is due only to the tension in the membrane. Thus, a membrane is similar to a string and, as was mentioned, is a two-dimensional version of a string. The reader is referred to Timoshenko, Young, and Weaver (1974) for the derivation of the membrane equation. Let $w(x, y, t)$ represent the displacement in the z direction of a membrane lying in the x–y plane at the point (x, y) and time t. The displacement is assumed to be small, with small slopes, and is perpendicular to the x–y plane. Let T be the tensile force per unit length of the membrane, assumed the same in all directions, and ρ be the mass per unit area of the membrane. Then, the equation for free vibration is given by

$$T\nabla^2 w(x, y, t) = \rho w_{tt}(x, y, t), \qquad x, y \in \Omega \tag{9.52}$$

where Ω denotes the region in the x–y plane occupied by the membrane. Here ∇^2 is the *Laplace operator*. In rectangular coordinates this operator has the form

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \tag{9.53}$$

The boundary conditions for the membrane must be specified along the shape of the boundary, not just at points, as in the case of the string. If the membrane is fixed or clamped at a segment of the boundary, then the deflection must be zero along that segment. If $\partial\Omega$ is the curve in the x–y plane corresponding to the edge of the membrane, i.e., the boundary of Ω, then the clamped boundary condition is denoted by

$$w(x, y, t) = 0, \qquad x, y \in \partial\Omega \tag{9.54}$$

If, for some segment of $\partial\Omega$, denoted by $\partial\Omega_1$, the membrane is free to deflect transversely, then there can be no force component in the transverse direction, and the boundary condition becomes

$$\frac{\partial w(x, y, t)}{\partial n} = 0, \qquad x, y \in \partial\Omega_1 \tag{9.55}$$

Here, $\partial w/\partial n$ denotes the derivative of $w(x, y, t)$ normal to the boundary in the reference plane of the membrane.

Example 9.5.1

Consider the vibration of a square membrane, as indicated in Figure 9.8, clamped at all of the edges. With $c^2 = T/\rho$, the equation of motion, [Equation (9.52)] becomes

$$c^2\left[\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right] = \frac{\partial^2 w}{\partial t^2}, \qquad x, y \in \Omega \tag{9.56}$$

Assuming that the solution separates, i.e., that $w(x, y, t) = X(x)Y(y)T(t)$, Equation (9.56) becomes

$$\frac{1}{c^2}\frac{\ddot{T}}{T} = \frac{X''}{X} + \frac{Y''}{Y} \tag{9.57}$$

Equation (9.57) implies that $\ddot{T}/(Tc^2)$ is a constant (recall the argument used in Section 9.2). Denote the constant by ω^2, so that

$$\frac{\ddot{T}}{Tc^2} = -\omega^2 \tag{9.58}$$

Then Equation (9.57) implies that

$$\frac{X''}{X} = -\omega^2 - \frac{Y''}{Y} \tag{9.59}$$

By the same argument used before, both X''/X and Y''/Y must be constant (that is, independent of t and x or y). Hence

$$\frac{X''}{X} = -\alpha^2 \tag{9.60}$$

and

$$\frac{Y''}{Y} = -\gamma^2 \tag{9.61}$$

where α^2 and γ^2 are constants. Equation (9.59) then yields

$$\omega^2 = \alpha^2 + \gamma^2 \tag{9.62}$$

These expressions result in two spatial equations to be solved

$$X'' + \alpha^2 X = 0 \tag{9.63}$$

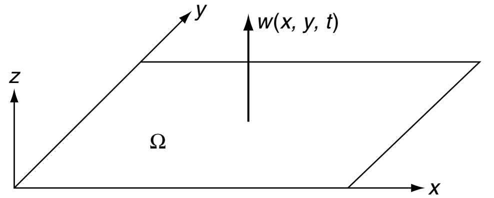

Figure 9.8 Square membrane illustrating vibration perpendicular to its surface.

which have a solution (A and B are constants of integration) of the form

$$X(x) = A \sin \alpha x + B \cos \alpha x \tag{9.64}$$

and

$$Y'' + \gamma^2 Y = 0 \tag{9.65}$$

which yields a solution (C and D are constants of integration) of the form

$$Y(y) = C \sin \gamma y + D \cos \gamma y \tag{9.66}$$

The total spatial solution is the product $X(x)Y(y)$, or

$$\begin{aligned} X(x)Y(y) = &A_1 \sin \alpha x \sin \gamma y + A_2 \sin \alpha x \cos \gamma y \\ &+ A_3 \cos \alpha x \sin \gamma y + A_4 \cos \alpha x \cos \gamma y \end{aligned} \tag{9.67}$$

Here, the constants A_i consist of the products of the constants in Equations (9.64) and (9.66) and are to be determined by the boundary and initial conditions.

Equation (9.67) can now be used with the boundary conditions to calculate the eigenvalues and eigenfunctions of the system. The clamped boundary condition, along $x = 0$ in Figure 9.8, yields

$$T(t)X(0)Y(y) = T(t)B(A_3 \sin \gamma y + A_4 \cos \gamma y) = 0$$

or

$$A_3 \sin \gamma y + A_4 \cos \gamma y = 0 \tag{9.68}$$

Now, Equation (9.68) must hold for any value of y. Thus, as long as γ is not zero (a reasonable assumption, since if it is zero the system has a rigid body motion), A_3 and A_4 must be zero. Hence, the spatial solution must have the form

$$X(x)Y(y) = A_1 \sin \alpha x \sin \gamma y + A_2 \sin \alpha x \cos \gamma y \tag{9.69}$$

Next, application of the boundary condition $w = 0$ along the line $x = 1$ yields

$$A_1 \sin \alpha \sin \gamma y + A_2 \sin \alpha \cos \gamma y = 0 \tag{9.70}$$

Factoring this expression yields

$$\sin \alpha (A_1 \sin \gamma y + A_2 \cos \gamma y) = 0 \tag{9.71}$$

Now, either $\sin \alpha = 0$ or, by the preceding argument, A_1 and A_2 must be zero. However, if A_1 and A_2 are both zero, the solution is zero. Hence, in order for a nontrivial solution to exist, $\sin \alpha = 0$, which yields

$$\alpha = n\pi, \qquad n = 1, 2, \ldots, \infty \tag{9.72}$$

Using the boundary condition $w = 0$ along the line $y = 1$ results in a similar procedure and yields

$$\gamma = m\pi, \qquad m = 1, 2, \ldots, \infty \tag{9.73}$$

Note that the possibility of $\gamma = \alpha = 0$ is not used because it was necessary to assume that $\gamma \neq 0$ in order to derive Equation (9.69). Equation (9.62) shows that the constant ω in the temporal equation must have the form

$$\begin{aligned} \omega_{mn} &= \sqrt{\alpha_n^2 + \gamma_m^2} \\ &= \pi\sqrt{m^2 + n^2}, \qquad m, n = 1, 2, 3, \ldots, \infty \end{aligned} \tag{9.74}$$

Thus, the eigenvalues and eigenfunctions for the clamped membrane are Equation (9.74) and $\{\mathit{sin}\ n\pi x$ and $\sin\ m\ \pi y\}$ respectively. The solution of Equation (9.56) becomes

$$\begin{aligned} w(x, y, t) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}(\sin m\pi x \sin n\pi y)\{[A_{mn}\sin[\sqrt{n^2 + m^2}c\pi t] \\ + B_{mn}\cos[\sqrt{n^2 + m^2}c\pi t]\} \end{aligned} \tag{9.75}$$

where A_{mn} and B_{mn} are determined by the initial conditions and the orthogonality of the eigenfunctions.

In progressing from the vibration of a string to considering the transverse vibration of a beam, the beam equation allowed for bending stiffness. In somewhat the same manner, a plate differs from a membrane because plates have bending stiffness. The reader is referred to Reismann (1988) or Sodel (1993) for a more detailed explanation and a precise derivation of the plate equation. Basically, the plate, like the membrane, is defined in a plane (x–y) with the deflection $w(x, y, t)$ taking place along the z axis perpendicular to the x–y plane. The basic assumption is again small deflections with respect to the thickness, h. Thus, the plane running through the middle of the plate is assumed not to deform during bending (called a *neutral plane*). In addition, normal stresses in the direction transverse to the plate are assumed to be negligible. Again, there is no thickness stretch. The displacement equation of motion for the free vibration of the plate is

$$-D_E\nabla^4 w(x, y, t) = \rho w_{tt}(x, y, t), \qquad x, y \in \Omega \tag{9.76}$$

where E again denotes the elastic modulus, ρ is the mass density (per unit area), and the constant D_E, the plate flexural rigidity, is defined in terms of Poisson's ratio ν and the plate thickness h as

$$D_E = \frac{Eh^3}{12(1 - \nu^2)} \tag{9.77}$$

The operator ∇^4, called the *biharmonic operator*, is a fourth-order operator, the exact form of which depends on the choice of coordinate systems. In rectangular coordinates, the biharmonic operator becomes

$$\nabla^4 = \frac{\partial^4}{\partial x^4} + 2\frac{\partial^4}{\partial x^2\partial y^2} + \frac{\partial^4}{\partial y^4} \tag{9.78}$$

The boundary conditions for a plate are a little more difficult to write, as their form, in some cases, also depends on the coordinate system in use.

Clamped edge. For a clamped edge, the deflection and normal derivative $\partial/\partial n$ are both zero along the edge:

$$w(x, y, t) = 0 \qquad \text{and} \qquad \frac{\partial w(x, y, t)}{\partial n} = 0, \qquad x, y \in \partial\Omega \tag{9.79}$$

Here, the normal derivative is the derivative of w normal to the neutral plane.

Simply supported. For a rectangular plate, the simply supported boundary conditions become

$$w(x, y, t) = 0 \qquad \text{along all edges}$$

$$\frac{\partial^2 w(x, y, t)}{\partial x^2} = 0 \qquad \text{along the edges } x = 0, x = \ell_1 \tag{9.80}$$

$$\frac{\partial^2 w(x, y, t)}{\partial y^2} = 0 \qquad \text{along the edges } y = 0, y = \ell_2 \tag{9.81}$$

where ℓ_1 and ℓ_2 are the lengths of the plate edges and the second partial derivatives reflect the normal strains along these edges.

9.6 LAYERED MATERIALS

The use of layered materials and composites in the design of modern structures has become very popular because of increased strength-to-weight ratios. The theory of vibration of layered materials is not as developed, but does offer some interesting design flexibility.

The transverse vibration of a three-layer beam consisting of a core between two faceplates, as indicated in Figure 9.9, is considered here. The layered beam consists of two faceplates of thickness h_1 and h_3, which are sandwiched around a core beam of thickness h_2. The distance between the center-lines of the two faceplates is denoted by d. The displacement equation of vibration becomes (see, for instance, Sun and Lu, 1995)

$$\frac{\partial^6 w(x, t)}{\partial x^6} - g(1+\beta)\frac{\partial^4 w(x, t)}{\partial x^4} + \frac{\rho}{D_E}\left[\frac{\partial^4 w}{\partial x^2 \partial t^2} - g\frac{\partial^2 w}{\partial t^2}\right] = 0, \qquad x \in (0, \ell) \tag{9.82}$$

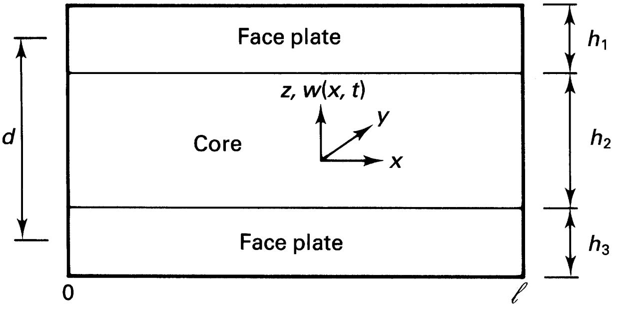

Figure 9.9 Cross-section of a three-layer beam for transverse vibration analysis.

where

$$
\begin{aligned}
g &= \frac{G}{h_2}\left[\frac{1}{E_1 h_1} + \frac{1}{E_3 h_3}\right] \\
G &= \text{shear modulus of the core} \\
E_i &= \text{Young's modulus of the } i\text{th face plate} \\
D_E &= \frac{E_1 h_1^3 + E_3 h_3^3}{12} \\
\beta &= \frac{d^2 E_1 h_1 E_3 h_3^3 (E_1 h_1 + E_3 h_3)}{D_E} \\
d &= h_2 + \frac{h_1 + h_3}{2} \\
\rho &= \text{mass per unit length of the entire structure}
\end{aligned}
$$

The boundary conditions are again not as straightforward as those of a simple beam. In fact, since the equation for free vibration contains six derivatives, there are six boundary conditions that must be specified. If the beam has both ends clamped, the boundary conditions are

$$
\begin{aligned}
w(0,t) = w(\ell,t) &= 0 \qquad \text{(zero displacement)} \\
w_x(0,t) = w_x(\ell,t) &= 0 \qquad \text{(zero rotation)} \\
w_{xxxx}(0,t) - g(1+\beta)w_{xx}(0,t) - \frac{\rho}{D_E} w_{tt}(0,t) &= 0 \\
w_{xxxx}(\ell,t) - g(1+\beta)w_{xx}(\ell,t) - \frac{\rho}{D_E} w_{tt}(\ell,t) &= 0 \qquad \text{(zero bending moments)}
\end{aligned}
\tag{9.83}
$$

Note that the form of this equation is quite different from all the other structures considered in this chapter. In all the previous cases, the equation for linear vibration can be written in the form

$$
\begin{aligned}
w_{tt}(\mathbf{x},t) + L_2 w(\mathbf{x},t) &= 0, \qquad \mathbf{x} \in \Omega \\
B(w)(\mathbf{x},t) &= 0, \qquad \mathbf{x} \in \partial\Omega
\end{aligned}
\tag{9.84}
$$

plus initial conditions, where Ω is a region in three-dimensional space bounded by $\partial\Omega$. Here, L_2 is a linear operator in the spatial variables only, $\boldsymbol{x}$ is a vector consisting of the spatial coordinates x, y, and z, and B represents the boundary conditions. As long as the vibration of a structure fits into the form of Equations (9.84) and the operator L_2 satisfies certain conditions (specified in Chapter 11), separation of variables can be used to solve the problem. However, Equation (9.82) is of the form

$$
\begin{aligned}
L_0 w_{tt}(\boldsymbol{x},t) + L_2 w(\mathbf{x},t) &= 0, \qquad \mathbf{x} \in \Omega \\
Bw(\mathbf{x},t) &= 0, \qquad \mathbf{x} \in \partial\Omega
\end{aligned}
\tag{9.85}
$$

where both L_0 and L_2 are linear operators in the spatial variables. Because of the presence of two operators it is not clear if separation of variables will work as a solution technique. The circumstances and assumptions required for separation of variables to work is the topic of the next chapter. In addition, classifications and further discussion of the operators are contained in Chapter 11.

The boundary conditions associated with the operator B in Equation (9.84) can be separated into two classes. Boundary conditions that arise clearly as a result of the geometry of the structure, such as Equation (9.47), are called *geometric boundary conditions*. Boundary conditions that arise by requiring a force (or moment) balance at the boundary are called *natural boundary conditions*. Equation (9.49) represents an example of a natural boundary condition.

9.7 VISCOUS DAMPING

Only viscous damping, introduced in Section 1.3, is considered in this section. The reason for this consideration is that viscous damping lends itself to analytical solutions for transient as well as steady state response by relatively simple techniques. While there is significant evidence indicating the inaccuracies of viscous damping models (for instance, Snowden, 1968), modeling dissipation as viscous damping represents a significant improvement over the conservative models given by Equations (9.84) or (9.85). In this section, several distributed-parameter models with viscous-type damping are presented.

First, consider the transverse free vibration of a membrane in a surrounding medium (such as air) furnishing resistance to the motion that is proportional to the velocity (i.e., viscous damping). The equation of motion is Equation (9.52) with the addition of a damping force. The resulting equation is

$$\rho w_{tt}(x, y, t) + \gamma w_t(x, y, t) - T\nabla^2 w(x, y, t) = 0, \qquad x, y \in \Omega \tag{9.86}$$

where ρ, T, and ∇^2 are as defined for Equation (9.53) and γ is the viscous damping coefficient. The positive constant γ reflects the proportional resistance to velocity. This system is subject to the same boundary conditions as those discussed in Section 9.5.

The solution method (separation of variables) outlined in example 9.5.1 works equally well for solving the damped membrane equation [Equation (9.86)]. The only change in the solution is that the temporal function $T(t)$ becomes an exponentially decaying sinusoid rather than a constant-amplitude sinusoid, depending on the relative size of γ.

External damping of the viscous type can also be applied to the model of the free flexural vibration of a plate. This problem has been considered by Murthy and Sherbourne (1972). They modeled the damped plate by

$$\rho w_{tt}(x, y, t) + \gamma w_t(x, y, t) + D_E\nabla^4 w(x, y, t) = 0, \qquad x, y, \in \Omega \tag{9.87}$$

subject to the same boundary conditions as Equation (9.86). Here, D_E, ρ, and ∇^4 are as defined for Equation (9.76), and γ again represents the constant viscous damping parameter. The plate boundary conditions given in Section 9.5 also apply to the damped plate equation [Equation (9.87)].

Finally, consider the longitudinal vibration of a bar subject to both internal and external damping. In this case, the equation of vibration for the bar in Figure 9.3 becomes

$$w_{tt}(x,t) + 2\left(\gamma - \beta\frac{\partial^2}{\partial x^2}\right)w_t(x,t) - \frac{EA}{\rho}w_{xx}(x,t) = 0, \qquad x \in \Omega \tag{9.88}$$

subject to the boundary conditions discussed in Section 9.3. The quantities E, A, and ρ are taken to be constant versions of the like quantities defined in Equation (9.29). The constant γ is again a viscous damping factor derived from an external influence, whereas the constant β is a viscous damping factor representing an internal damping mechanism.

Note that the internal model is slightly more complicated than the other viscous damping models considered in this section because of the inclusion of the second spatial derivative. Damping models involving spatial derivatives can cause difficulties in computing analytical solutions and are discussed in Sections 10.4 and 11.6. In addition, damping terms can alter the boundary conditions.

The general model for vibration of distributed-parameter systems with viscous damping can be written as

$$\begin{aligned} L_0 w_{tt}(\mathbf{x},t) + L_1 w_t(\mathbf{x},t) + L_2 w(\mathbf{x},t) &= 0, \qquad \mathbf{x} \in \Omega \\ Bw(\mathbf{x},t) &= 0, \qquad \mathbf{x} \in \partial\Omega \end{aligned} \tag{9.89}$$

plus appropriate initial conditions. Here, the operators B, L_0, and L_2 are as defined for Equation (9.85), and the operator L_1 is exemplified by the models illustrated in this section. The operator L_0 is called the mass operator, the operator L_1 is called the damping operator, and the operator L_2 is called the stiffness operator. As illustrated by the examples, the operator L_0 is often the identity operator. The properties of these operators, the nature of the solutions of Equations (9.84) and (9.89), and their relationship to the vibration problem are topics of the next three chapters.

CHAPTER NOTES

This chapter presents a brief introduction to the linear vibration of distributed-parameter systems without regard for mathematical rigor or a proper derivation of the governing equations. The chapter is intended to review and familiarize the reader with some of the basic structural elements used as examples in the study of distributed-parameter systems and points out the easiest and most commonly used method of solution – separation of variables.

Section 9.2 introduces the classic vibrating string. The derivation and solution of the string equation can be found in almost any text on vibration, partial differential equations, or applied mathematics. The vibration of bars covered in Section 9.3 is almost as common and can again be found in almost any vibration text. An excellent detailed derivation of most of the equations can be found in Magrab (1979) and more basic derivations can be found in Inman (2001) or any of the other excellent introductory texts (such as Rao, 2004). The material on membranes and plates of Section 9.5 is also very standard and can be found in most advanced texts, such as Meirovitch (1967, 1997, 2001). Ventsel and Krauthammer (2001) present a complete derivation of the thin plate equations used here. A classic reference for plates is Sodel (1993). The material on layered structures of Section 9.6 is nonstandard, but such materials have made a significant impact in engineering design and should be considered. Sun and Lu's (1995) book gives a list of useful papers in this area. Blevins (2001) is an excellent reference and tabulates natural frequencies and mode shapes for a variety of basic elements (strings, bars, beams, and plates in various configurations). Elishakoff (2005) gives the solutions of several unusual configurations.

Section 9.7 introduces some simple viscous damping models for distributed-parameter systems. Information on such models in the context of transient vibration analysis is difficult to come by. The majority of work on damping models centers on the steady state forced response of such systems and presents a very difficult problem. Snowden (1968) and Nashif, Jones, and Henderson (1985) present an alternative view on damping and should be consulted for further reading. Banks, Wang, and Inman (1994) discuss damping in beams.

As indicated, most of the material in this chapter is standard. However, this chapter does include some unusual boundary conditions representing lumped-parameter elements appended to distributed-parameter structures. These configurations are very important in vibration design and control. The texts of Gorman (1975) and of Blevins (2001) tabulates the natural frequencies of such systems. Such configurations are analyzed in Section 12.4.

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PROBLEMS

9.1 Consider the bar of Section 9.3. Using the method of separation of variables, calculate the natural frequencies if the bar is clamped at one end and connected to a linear spring of stiffness k at the other end.

9.2 Consider the string equation of Section 9.2, with clamped boundaries at each end. At the midpoint, $x = \ell/2$, the density changes from ρ_1 to ρ_2, but the tension T remains constant. Derive the characteristic equation.

9.3 Calculate the natural frequencies of vibration of the Euler–Bernoulli beam clamped at both ends.

9.4 Consider a nonuniform bar described by

$$\frac{\partial}{\partial x}[EA(x)u_x] = m(x)u_{tt} \text{ on } (0, \ell)$$

with

$$EA(a) = 2EA_0\left(1 - \frac{x}{\ell}\right) \qquad \text{and} \qquad m(x) = 2m_0\left(1 - \frac{x}{\ell}\right)$$

fixed at 0 and free at ℓ. Calculate the free vibration. What are the first three eigenvalues?

9.5 Solve the Timoshenko beam equation [Equation (9.45)]. Assume that the cross-section remains constant and eliminate ψ from the equations to produce a single equation in the displacement. Compare your result with the Euler–Bernoulli equation and identify the shear deformation term and rotary inertia term. What can you conclude?

9.6 Verify the orthogonality condition for the set of functions

$$\left\{\sin\frac{n\pi x}{\ell}\right\}$$

on the interval $[0, \ell]$.

9.7 Calculate the natural frequencies of the system in Figure 9.4 and compare them with those of example 9.3.1.

9.8 Calculate the solution of the internally and externally damped bar given by Equation (9.88) with a clamped boundary at $x = 0$ and a free boundary at $x = \ell$.

9.9 Are there values of the parameters γ and ρ in Equation (9.88) for which the damping for one or more of the terms $T_n(t)$ is zero? A minimum? (Use the clamped boundary conditions of problem 9.8.)

9.10 Consider the damped membrane of Equation (9.86) with clamped boundary conditions. Calculate values of γ, ρ, and T such that the temporal part, $T(t)$, of the separation of variables solution does not oscillate.

9.11 Compute the natural frequencies of a beam clamped at the right end (0) with a tip mass of value M at the left end.

9.12 Compute the characteristic equation of a shaft with a disc of inertia I at each end.