10
Formal Methods of Solution

10.1 INTRODUCTION

This chapter examines various methods of solving for the vibrational response of the distributed-parameter systems introduced in the previous chapter. As in finite-dimensional systems, the response of a given system is made up of two parts: the transient, or free, response, and the steady state, or forced, response. In general the steady state response is easier to calculate, and in many cases the steady state is all that is necessary. The focus in this chapter is the free response. The forced response is discussed in more detail in Chapter 12.

Several approaches to solving distributed-parameter vibration problems are considered. The formal notion of an operator and the eigenvalue problem associated with the operator are introduced. The traditional separation of variables method used in Chapter 9 is compared with the eigenvalue problem. The eigenfunction expansion method is introduced and examined for systems including damping. Transform methods and integral formulations in the form of Green's functions are also introduced in less detail.

10.2 BOUNDARY VALUE PROBLEMS AND EIGENFUNCTIONS

As discussed in Section 9.6, a general formulation of the undamped boundary value problems presented in Chapter 9 can be written as (the subscript on L is dropped here for notational ease)

$$\begin{aligned} w_{tt}(\mathbf{x},t) + Lw(\mathbf{x},t) = 0, &\qquad \mathbf{x}\in\Omega \qquad \text{for } t>0 \\ Bw = 0, &\qquad \mathbf{x}\in\partial\Omega \qquad \text{for } t>0 \\ w(\mathbf{x},0) = w_0(\mathbf{x}), &\qquad w_t(\mathbf{x},0) = \dot{w}_0(\mathbf{x}) \qquad \text{at } t=0 \end{aligned} \tag{10.1}$$

where $w(\mathbf{x}, t)$ is the deflection, $\mathbf{x}$ is a three-dimensional vector of spatial variables, and Ω is a bounded region in three-dimensional space with boundary $\partial\Omega$. The operator L is

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a differential operator of spatial variables only. For example, for the longitudinal vibration of a beam (string or rod) the operator L has the form

$$L = -\alpha \frac{\partial^2}{\partial x^2} \tag{10.2}$$

where α is a constant. An *operator*, or *transformation*, is a rule that assigns, to each function $w(\mathbf{x}, t)$ belonging to a certain class, another function ($-aw_{xx}$ in the case of the string operator) belonging to another, perhaps different, class of functions. Note that a matrix satisfies this definition. B is an operator representing the boundary conditions as given, for example, by Equations (9.2). As indicated previously, the equations of (10.1) define a boundary value problem. A common method of solving (10.1) is to use separation of variables, as illustrated by the examples in Chapter 9. As long as the operator L does not depend on time, and if L satisfies certain other conditions (discussed in the next chapter), this method will work.

In many situations, the separation of variables approach yields an infinite set of functions of the form $a_n(t)\phi_n(\mathbf{x})$ that are solutions of Equations (10.1). The most general solution is then the sum, i.e.,

$$w(\mathbf{x}, t) = \sum_{n=1}^{\infty} a_n(t)\phi_n(\mathbf{x}) \tag{10.3}$$

A related method, modal analysis, also uses these functions and is described in the next section.

Similar to the eigenvectors of a matrix, some operators have eigenfunctions. A nonzero function $\phi(\mathbf{x})$ that satisfies the relationships

$$L\phi(\mathbf{x}) = \lambda\phi(\mathbf{x}), \qquad \mathbf{x} \in \Omega$$
$$B\phi(\mathbf{x}) = 0, \qquad \mathbf{x} \in \partial\Omega$$

is called an *eigenfunction* of the operator L with boundary conditions B. The scalar λ (possibly complex) is called an *eigenvalue* of the operator L with respect to the boundary conditions B. In some cases the boundary conditions are not present, as in the case of a matrix, and in some cases the boundary conditions are contained in the domain of the operator L. The *domain* of the operator L, denoted by $D(L)$, is the set of all functions $u(x)$ for which Lu is defined and of interest.

To see the connection between separation of variables and eigenfunctions, consider substitution of the assumed separated form $w(\mathbf{x}, t) = a(t)\phi(\mathbf{x})$ into Equations (10.1). This yields

$$\frac{\ddot{a}(t)}{a(t)} = \frac{L\phi(\mathbf{x})}{\phi(\mathbf{x})}, \qquad \mathbf{x} \in \Omega, \qquad t > 0 \tag{10.4}$$

$$a(t)B\phi(\mathbf{x}) = 0, \qquad \mathbf{x} \in \partial\Omega, \qquad t > 0 \tag{10.5}$$

$$a(0)\phi(\mathbf{x}) = w_0(\mathbf{x}), \qquad \dot{a}(0)\phi(\mathbf{x}) = \dot{w}_0(\mathbf{x}) \tag{10.6}$$

As before, Equation (10.4) implies that each side is constant, so that

$$L\phi(\mathbf{x}) = \lambda\phi(\mathbf{x}), \qquad \mathbf{x} \in \Omega \tag{10.7}$$

where λ is a scalar. In addition, note that, since $a(t) \neq 0$ for all t, Equation (10.5) implies that

$$B\phi(\mathbf{x}) = 0, \qquad x \in \partial\Omega, \qquad t > 0 \tag{10.8}$$

Equations (10.7) and (10.8) are, of course, a statement that $\phi(\mathbf{x})$ and λ constitute an eigenfunction and eigenvalue of the operator L.

Example 10.2.1

Consider the operator formulation of the longitudinal bar equation presented in Section 9.3. The form of the beam operator is

$$L = -\alpha \frac{\partial^2}{\partial x^2}, \qquad x \in (0, \ell)$$

with boundary conditions ($B = 1$ at $x = 0$ and $B = \partial/\partial x$ at $x = \ell$)

$$\phi(0) = 0 \quad \text{(clamped end)} \qquad \text{and} \qquad \phi_x(\ell) = 0 \quad \text{(free end)}$$

and where $\partial\Omega$ consists of the points $x = 0$ and $x = \ell$. Here, the constant a represents the physical parameters of the beam, i.e., $\alpha = EA/\rho$. The eigenvalue problem $L\phi = \lambda\phi$ becomes

$$-\alpha\phi_{xx} = \lambda\phi$$

or

$$\phi_{xx} + \frac{\lambda}{\alpha}\phi = 0$$

This last expression is identical to Equation (9.6) and the solution is

$$\phi(x) = A_1 \sin\left(\sqrt{\frac{\lambda}{\alpha}}x\right) + A_2 \cos\left(\sqrt{\frac{\lambda}{\alpha}}x\right)$$

where A_1 and A_2 are constants of integration. Using the boundary conditions yields

$$0 = \phi(0) = A_2 \qquad \text{and} \qquad 0 = \phi_x(\ell) = A_1\sqrt{\frac{\lambda}{\alpha}} \cos\sqrt{\frac{\lambda}{\alpha}}\ell$$

This requires that $A_2 = 0$ and

$$A_1 \cos\sqrt{\frac{\lambda}{\alpha}}\ell = 0$$

Since A_1 cannot be zero,

$$\sqrt{\frac{\lambda}{\alpha}}\ell = \frac{n\pi}{2}$$

for all odd integers n. Thus, λ depends on n and

$$\lambda_n = \frac{\alpha n^2 \pi^2}{4\ell^2}, \qquad n = 1, 3, 5, \ldots, \infty$$

Thus, there are many eigenvalues λ, denoted now by λ_n, and many eigenfunctions ϕ, denoted by ϕ_n. The eigenfunctions and eigenvalues of the operator L are given by the sets

$$\{\phi_n(x)\} = \left\{A_n \sin\frac{n\pi x}{2\ell}\right\} \qquad \text{and} \qquad \{\lambda_n\} = \left\{\frac{\alpha n^2\pi^2}{4\ell^2}\right\}, \; n \text{ odd}$$

respectively. Note that, as in the case of a matrix eigenvector, eigenfunctions are determined only to within a multiplicative constant (A_n in this case).

Comparison of the eigenfunctions of the operator for the beam with the spatial functions calculated in Chapter 9 shows that the eigenfunctions of the operator correspond to the mode shapes of the structure. This correspondence is exactly analogous to the situation for the eigenvectors of a matrix.

10.3 MODAL ANALYSIS OF THE FREE RESPONSE

The eigenfunctions associated with the string equation are shown in the example of Section 9.2 to be the mode shapes of the string. Also, by using the linearity of the equations of motion, the solution is given as a summation of mode shapes. This summation of mode shapes, or eigenfunctions, given in Equation (9.16), constitutes the *eigenfunction expansion or modal analysis* of the solution and provides an alternative point of view to the separation of variables technique.

First, as in the case of eigenvectors of a matrix, eigenfunctions are conveniently normalized to fix a value for the arbitrary constant. To this end, let the eigenfunctions of interest be denoted by $A_n\phi_n(\mathbf{x})$. If the constants A_n are chosen such that

$$\int_\Omega A_n^2\phi_n(\mathbf{x})\phi_n(\mathbf{x})\,d\Omega = 1 \tag{10.9}$$

then the eigenfunctions $\Theta_n = A_n\phi_n$ are said to be *normalized*, or normal. If, in addition, they satisfy

$$\int_\Omega \Theta_n\Theta_m d\Omega = \delta_{mn} \tag{10.10}$$

the eigenfunctions are said to be *orthonormal*, exactly analogous to the eigenvector case. The method of modal analysis assumes that the solution of Equations (10.1) can be represented as the series

$$w(\mathbf{x}, t) = \sum_{n=1}^{\infty} a_n(t)\Theta_n(\mathbf{x}) \tag{10.11}$$

where $\Theta_n(\mathbf{x})$ are the normalized eigenfunctions of the operator L. Substitution of Equation (10.11) into Equations (10.1), multiplying by $\Theta_m(\mathbf{x})$, and integrating (assuming uniform convergence) over the domain Ω reduces Equations (10.1) to an infinite set of uncoupled ordinary differential equations of the form

$$\ddot{a}_n(t) + \lambda_n a_n(t) = 0, \qquad n = 1, 2, \ldots, \infty \tag{10.12}$$

Equation (10.12) can then be used along with the appropriate initial conditions to solve for each of the temporal functions. Here, λ_n is the eigenvalue associated with the nth mode, so that

$$\int_{\Omega} \Theta_n L \Theta_n d\Omega = \lambda_n \int_{\Omega} \Theta_n \Theta_n d\Omega = \lambda_n \tag{10.13}$$

where Equation (10.7) and (10.10) are used to evaluate the integral.

Example 10.3.1

Consider the transverse vibration of a Euler–Bernoulli beam with hinged boundary conditions. Calculate the eigenvalues and eigenfunctions for the associated operator.

The stiffness operator for constant mass, cross-sectional area, and area moment of inertia is given by (see Equation 9.46)

$$L = \frac{EI}{m}\frac{\partial^4}{\partial x^4} = \beta \frac{\partial^4}{\partial x^4}$$
$$\Theta(0) = \Theta_{xx}(0) = 0$$
$$\Theta(\ell) = \Theta_{xx}(\ell) = 0$$

The eigenvalue problem $Lu = \lambda u$ then becomes

$$\beta \Theta_{xxxx} = \lambda \Theta$$

which has a solution of the form

$$\Theta(x) = C_1 \sin \mu x + C_2 \cos \mu x + C_3 \sinh \mu x + C_4 \cosh \mu x$$

where $\mu^4 = \lambda/\beta$. Applying the four boundary conditions to this expression yields the four simultaneous equations

$$\begin{aligned}
\Theta(0) &= C_2 + C_4 = 0 \\
\Theta_{xx}(0) &= -C_2 + C_4 = 0 \\
\Theta(L) &= C_1 \sin \mu\ell + C_2 \cos \mu\ell + C_3 \sin \mu\ell + C_4 \cos \mu\ell = 0 \\
\Theta_{xx}(\ell) &= -C_1 \sin \mu\ell - C_2 \cos \mu\ell + C_3 \sin \mu\ell + C_4 \cos \mu\ell = 0
\end{aligned}$$

These four equations in the four unknown constants C_i can be solved by examining the matrix equation

$$\begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & -1 & 0 & 1 \\ \sin \mu\ell & \cos \mu\ell & \sinh \mu\ell & \cosh \mu\ell \\ -\sin \mu\ell & -\cos \mu\ell & \sinh \mu\ell & \cosh \mu\ell \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Recall from Chapter 3 that, in order for a nontrivial vector $\mathbf{c} = [C_1 \quad C_2 \quad C_3 \quad C_4]^T$ to exist, the coefficient matrix must be singular. Thus, the determinant of the coefficient matrix must be zero. Setting the determinant equal to zero yields the characteristic equation

$$4 \sin(\mu\ell) \sinh(\mu\ell) = 0$$

This, of course, can be true only if $\sin(\mu\ell) = 0$, leading to

$$\mu = \frac{n\pi}{\ell}, \qquad n = 1,\ 2, \ldots, \infty$$

Here, $n = 0$ is excluded because it results in the trivial solution. In terms of the physical parameters of the structure, the eigenvalues become (here n is an integer and m is the mass per unit length of the beam)

$$\lambda_n = \frac{n^4\pi^4 EI}{m\ell^4}$$

Solving for the four constants C_i yields $C_2 = C_3 = C_4 = 0$ and that is C_1 arbitrary. Hence, the eigenfunctions are of the form

$$\left[A_n \sin\left(\frac{n\pi x}{\ell}\right)\right]$$

The arbitrary constants A_n can be fixed by normalizing the eigenfunctions

$$\int_0^{\ell} A_n^2 \sin^2\left(\frac{n\pi x}{\ell}\right) dx = 1$$

so that $A_n^2 \ell/2 = 1$, or $A_n = \sqrt{2/\ell}$. Thus, the normalized eigenfunctions are the set

$$\{\Theta_n\} = \left\{\sqrt{\frac{2}{\ell}} \sin\left(\frac{n\pi x}{\ell}\right)\right\}_{n=1}^{\infty}$$

Hence, the temporal coefficient in the series expansion of the solution (10.11) will be determined from the initial conditions and the finite number of equations

$$\ddot{a}_n(t) + \frac{n^4\pi^4 EI}{m\ell^4} a_n(t) = 0, \qquad n = 1, \ldots, \infty$$

Equation (10.11) then yields the total solution.

10.4 MODAL ANALYSIS IN DAMPED SYSTEMS

As in the matrix case for lumped-parameter systems, the method of modal analysis (and separation of variables) can still be used for certain types of viscous damping modeled in a distributed structure. Systems that can be modeled by partial differential equations of the form

$$w_{tt}(\mathbf{x}, t) + L_1 w_t(\mathbf{x}, t) + L_2 w(\mathbf{x}, t) = 0, \qquad \mathbf{x} \in \Omega \tag{10.14}$$

(where L_1 and L_2 are operators, with similar properties to L and such that L_1 and L_2 have the same eigenfunctions) can be solved by the method of modal analysis illustrated in Equation (10.11) and example 10.3.1. Section 9.7 lists some examples.

To see this solution method, let L_1 have eigenvalues $\lambda_n^{(1)}$ and L_2 have eigenvalues $\lambda_n^{(2)}$. Substitution of Equation (10.11) into Equation (10.14) then yields (assuming convergence)

$$\sum_{n=1}^{\infty}\left[\ddot{a}_n\Theta_n(\mathbf{x})+\lambda_n^{(1)}\dot{a}_n\Theta_n(\mathbf{x})+\lambda_n^{(2)}a_n\Theta_n(\mathbf{x})\right]=0 \tag{10.15}$$

Multiplying by $\Theta_n(\mathbf{x})$, integrating over Ω, and using the orthogonality conditions (10.10) yields the decoupled set of n ordinary differential equations

$$\ddot{a}_n(t)+\lambda_n^{(1)}\dot{a}_n(t)+\lambda_n^{(2)}a_n(t)=0, \qquad n=1,2,\ldots,\infty \tag{10.16}$$

subject to the appropriate initial conditions.

The actual form of damping in distributed-parameter systems is not always clearly known. In fact, the form of L_1 is an elusive topic of current research and several texts (see, for instance, Nashif, Jones, and Henderson, 1985, or Sun and Lu, 1995). Often, the damping is modeled as being proportional, i.e., $L_1=\alpha I+\beta L_2$, where α and β are arbitrary scalars and L_1 satisfies the same boundary conditions as L_2. In this case, the eigenfunctions of L_1 are the same as those of L_2. Damping is often estimated using equivalent viscous proportional damping of this form as an approximation.

Example 10.4.1

As an example of a proportionally damped system, consider the transverse free vibration of a membrane in a surrounding medium, such as a fluid, providing resistance to the motion that is proportional to the velocity. The equation of motion given by Equation (9.86) is Equation (10.14), with

$$L_1=2\frac{\gamma}{\rho}$$
$$L_2=-\frac{T}{\rho}\nabla^2$$

where γ, T, ρ, and ∇^2 are as defined for Equation (9.86). The position $\mathbf{x}$ in this case is the vector $[x \quad y]$ in two-dimensional space. If $\lambda_1^{(2)}$ is the first eigenvalue of L_2, then the solutions to Equation (10.16) are of the form

$$a_n(t)=\mathrm{e}^{-\frac{\gamma}{\rho}t}\left[A_n\sin\sqrt{\lambda_n^{(2)}-\frac{\gamma^2}{\rho^2}}\,t+B_n\cos\sqrt{\lambda_n^{(2)}-\frac{\gamma^2}{\rho^2}}\,t\right]$$

where A_n and B_n are determined by the initial conditions (see Section 11.9).

Not all damped systems have this type of damping. Systems that have proportional damping are called *normal mode systems*, since the eigenfunctions of the operator L_2 serve to 'decouple' the system. Decouple, as used here, refers to the fact that Equations(10.16) depends only on n and not on any other index. This topic is considered in more detail in Section 11.9.

10.5 TRANSFORM METHODS

An alternative to using separation of variables and modal analysis is to use a transform to solve for the vibrational response. As with the Laplace transform method used on the temporal variable in state-space analysis for lumped-parameter systems, a Laplace transform can also be used in solving Equations (10.1). In addition, a Fourier transform can be used on the spatial variable to calculate the solution. These methods are briefly mentioned here. The reader is referred to a text such as Churchill (1972) for a rigorous development.

The Laplace transform taken on the temporal variable of a partial differential equation can be used to solve for the free or forced response of Equations (10.1) and (10.14). This method is best explained by considering an example.

Consider the vibrations of a beam with constant force F_0 applied to one end and fixed at the other. Recall that the equation for longitudinal vibration is

$$w_{tt}(x,t) = \alpha^2 w_{xx}(x,t) \tag{10.17}$$

with boundary conditions

$$w(0,t) = 0, \qquad EAw_x(\ell,t) = F_0\delta(t) \tag{10.18}$$

Here, $a^2 = EA/\rho$, as defined in Section 9.3. Assuming that the initial conditions are zero, the Laplace transform of Equation (10.17) yields

$$s^2 W(x,s) - \alpha^2 W_{xx}(x,s) = 0 \tag{10.19}$$

and the Laplace transform of Equation (10.18) yields

$$\begin{aligned} W_x(\ell,s) &= \frac{F_0}{EAs} \\ W(0,s) &= 0 \end{aligned} \tag{10.20}$$

Here, W denotes the Laplace transform of w. The solution of Equation (10.19) is of the form

$$W(x,s) = A_1 \sinh\frac{sx}{\alpha} + A_2 \cosh\frac{sx}{\alpha}$$

Applying the boundary condition at $x = 0$, gives $A_2 = 0$. Differentiating with respect to x and taking the Laplace transform yields the boundary condition at $x = \ell$. The constant A_1 is then determined to be

$$A_1 = \left(\frac{\alpha F_0}{EA}\right)\left(\frac{1}{s^2\cosh(s\ell/\alpha)}\right)$$

The solution in terms of the transform variable s then becomes

$$W(x,s) = \frac{\alpha F_0 \sinh(sx/\alpha)}{EAs^2\cosh(s\ell/\alpha)} \tag{10.21}$$

By taking the inverse Laplace transform of Equation (10.21) using residue theory, the solution in the time domain is obtained. The inverse is given by Churchill (1972) to be

$$w(x,t)=\frac{F_0}{E}x+\left(\frac{8\ell}{\pi}\right)^2\sum_{n=1}^{\infty}\frac{(-1)^n}{(2n-1)^2}\sin\frac{(2n-1)\pi x}{2\ell}\cos\frac{(2n-1)\pi a t}{2\ell} \tag{10.22}$$

A text on transforms should be consulted for the details. Basically, the expansion comes from the zeros in the complex plane of $s^2\cosh(s\ell/a)$, i.e., the poles of $W(x,s)$.

This same solution can also be obtained by taking the finite Fourier sine transform of Equations (10.17) and (10.18) on the spatial variable x rather than the Laplace transform of the temporal variable (see, for instance, Meirovitch, 1967). Usually, transforming the spatial variable is more productive because the time dependence is a simple initial value problem.

When boundary conditions have even-order derivatives, a finite sine transformation (Fourier transform) is appropriate. The sine transform is defined by

$$W(n,t)=\int_0^{\ell} w(x,t)\sin\frac{n\pi x}{\ell}\,dx \tag{10.23}$$

Note here that the transform in this case is over the spatial variable.

Again, the method is explained by example. To that end, consider the vibration of a string clamped at each end and subject to nonzero initial velocity and displacement, i.e.,

$$\begin{gathered} w_{xx}=\frac{1}{c^2}w_{tt}(x,t) \\ w(0,t)=w(\ell,t)=0, \qquad w(x,0)=f(x), \qquad w_t(x,0)=g(x) \end{gathered} \tag{10.24}$$

The finite sine transform of the second derivative is

$$W_{xx}(n,t)=\frac{n\pi}{\ell}\left|(-1)^{n+1}W(\ell,t)+W(0,t)\right|-\left(\frac{n\pi}{\ell}\right)^2 W(n,t) \tag{10.25}$$

which is calculated from integration by parts of Equation (10.23). Substitution of the boundary conditions yields the transformed string equation

$$W_{tt}(n,t)+\left(\frac{n\pi}{\ell}\right)^2 W(n,t)=0 \tag{10.26}$$

This equation is subject to the transform of the initial conditions, which are

$$W(n,0)=\int_0^{\ell} f(x)\sin\frac{n\pi x}{\ell}\,dx \tag{10.27}$$

and

$$W_t(n,0)=\int_0^{\ell} g(x)\sin\frac{n\pi x}{\ell}\,dx$$

Thus

$$W(n,t)=W(n,0)\cos\frac{n\pi c t}{\ell}+W_t(n,0)\frac{\ell}{n\pi c}\sin\frac{n\pi c t}{\ell} \tag{10.28}$$

Again, Equation (10.28) has to be inverted to obtain the solution $w(x, t)$. The inverse finite Fourier transform is given by

$$w(x, t) = \frac{2}{\ell} \sum_{n=1}^{\infty} W(n, t) \sin\left(\frac{n\pi x}{\ell}\right) \tag{10.29}$$

so that

$$w(x, t) = \frac{2}{\ell} \sum_{n=1}^{\infty} \left[\left\{ W(n, 0) \cos\left(\frac{n\pi ct}{\ell}\right) + \frac{W_t(n, 0)\ell}{n\pi c} \sin\left(\frac{n\pi ct}{\ell}\right) \right\} \sin \frac{n\pi x}{\ell} \right] \tag{10.30}$$

Transform methods are attractive for problems defined over infinite domains and for problems with odd boundary conditions. The transform methods yield a quick 'solution' in terms of the transformed variable. However, the inversion back into the physical variable can be difficult and may require as much work as using separation of variables or modal analysis. However, in some instances, the only requirement may be to examine the solution in its transformed state, such as is done in Section 8.5.

10.6 GREEN'S FUNCTIONS

Yet another approach to solving the free vibration problem is to use the integral formulation of the equations of motion. The basic idea here is that the free response is related to the eigenvalue problem

$$\begin{aligned} Lw &= \lambda w \\ Bw &= 0 \end{aligned} \tag{10.31}$$

where L is a differential operator and B represents the boundary conditions. The inverse of this operator will also yield information about the free vibrational response of the structure. If the inverse of L exists, Equation (10.31) can be written as

$$L^{-1} w = \frac{1}{\lambda} w \tag{10.32}$$

where L^{-1} is the inverse of the differential operator or an integral operator.

The problem of solving for the free vibration of a string fixed at both ends by working essentially with the inverse operator is approached in this section. This is done by introducing the concept of a Green's function. To this end, consider again the problem of a string fixed at both ends and deformed from its equilibrium position. This time, however, instead of looking directly at the vibration problem, the problem of determining the static deflection of the string owing to a transverse load concentrated at a point is first examined. This related problem is called the *auxiliary problem*. In particular, if the string is subject to a point load of unit value at x_0, which is somewhere in the interval (0,1), the equation of the deflection $w(x)$ for a string of tension T is

$$-T \frac{d^2 w(x)}{dx^2} = \delta(x - x_0) \tag{10.33}$$

where $\delta(x-x_0)$ is the Dirac delta function. The delta function is defined by

$$\delta(x-x_0)\begin{cases} 0, & x \neq x_0 \\ \infty, & x = x_0 \end{cases} \tag{10.34}$$

and

$$\int_0^1 \delta(x-x_0)dx = \begin{cases} 0 & \text{if } x_0 \text{ is not in } (0,1) \\ 1 & \text{if } x_0 \text{ is in } (0,1) \end{cases} \tag{10.35}$$

If $f(x)$ is a continuous function, then it can be shown that

$$\int_0^1 f(x)\delta(x-x_0)dx = f(x_0) \tag{10.36}$$

for x_0 in (0,1). Note that the Dirac delta function is not really a function in the strict mathematical sense (see, for instance, Stakgold, 1979).

Equation (10.33) can be viewed as expressing the fact that the force causing the deflection is applied only at the point x_0. Equation (10.33) plus boundary conditions is now viewed as the auxiliary problem of finding a function $g(x, x_0)$, known as *Green's function* for the operator $L = -Td^2/dx^2$, with boundary conditions $g(0, x_0) = 0$ and $g(1, x_0) = 0$. In more physical terms, $g(x, x_0)$ represents the deflection of the string from its equilibrium position at point x owing to a unit force applied at point x_0. Green's function thus defined is also referred to as an *influence function*. The following example is intended to clarify the procedure for calculating a Green's function.

Example 10.6.1

Calculate Green's function for the string of Figure 10.1. Green's function is calculated by solving the equation on each side of the point x_0 and then matching up the two solutions. Thus, since $g'' = 0$ for all x not equal to x_0, integrating yields

$$g(x, x_0) = \begin{cases} Ax + B, & 0 < x < x_0 \\ Cx + D, & x_0 < x < 1 \end{cases}$$

where A, B, C, and D are constants of integration. Applying the boundary condition at $x = 0$ yields

$$g(0, x_0) = 0 = B$$

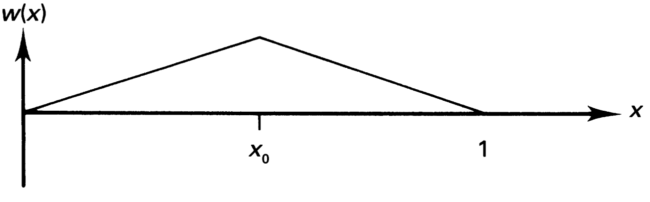

Figure 10.1 Statically deflected string fixed at both ends.

and the boundary condition at 1 yields

$$g(1, x_0) = 0 = C + D$$

Hence, Green's function becomes

$$g(x, x_0) = \begin{cases} Ax, & 0 < x < x_0 \\ C(x-1), & x_0 < x < 1 \end{cases}$$

Since the string does not break at x_0, $g(x, x_0)$ must be continuous at x_0, and this allows determination of one more constant. In particular, this continuity condition requires that

$$Ax_0 = C(x_0 - 1)$$

Green's function now becomes

$$g(x, x_0) \begin{cases} Ax & 0 < x < x_0 \\ A\dfrac{x_0(x-1)}{x_0 - 1}, & x_0 < x < 1 \end{cases}$$

The remaining constant, A, can be evaluated by considering the magnitude of the applied force required to produce the deflection. In this case, a unit force is applied, so that integration of Equation (10.33) along a small interval containing x_0, say, $x_0 - \varepsilon < x < x_0 + \varepsilon$, yields

$$\int_{x_0-\varepsilon}^{x_0+\varepsilon} \frac{d^2 g}{dx^2}\, dx = -\frac{1}{T} \int_{x_0-\varepsilon}^{x_0+\varepsilon} \delta(x - x_0)\, dx$$

or

$$\left.\frac{dg}{dx}\right|_{x_0-\varepsilon}^{x_0+\varepsilon} = -\frac{1}{T}$$

Denoting the derivative by a subscript and expanding yields

$$g_x(x_0 + \varepsilon, x_0) - g_x(x_0 - \varepsilon, x_0) = -\frac{1}{T}$$

This last expression is called a *jump discontinuity* in the derivative. Upon evaluating the derivative, the above expression becomes

$$A\frac{x_0}{x_0 - 1} - A = -\frac{1}{T}$$

Solving this for the value of A yields

$$A = \frac{1 - x_0}{T}$$

Green's function, with all the constants of integration evaluated, is thus

$$g(x, x_0) = \begin{cases} \dfrac{(1 - x_0)x}{T}, & 0 < x < x_0 \\ \dfrac{(1 - x)x_0}{T}, & x_0 < x < 1 \end{cases}$$

Green's function actually defines the inverse operator (when it exists) of the differential operator L and can be used to solve for the forced response of the string. Consider the equations (for the string operator of the example)

$$\begin{aligned} Lu &= f(x) \\ Bu &= 0 \end{aligned} \tag{10.37}$$

where $f(x)$ is a piecewise continuous function, and L is a differential operator that has an inverse. Let G denote the operator defined by

$$Gf(x) = \int_0^1 g(x, x_0) f(x_0)\, dx_0$$

The operator G defined in this way is called an *integral operator*. Note that the function

$$u(x) = \int_0^1 g(x, x_0) f(x_0)\, dx_0 \tag{10.38}$$

satisfies Equation (10.37), including the boundary conditions, which follows from a straightforward calculation. Equation (10.38) can also be written as

$$u(x) = \int_0^{x-\varepsilon} g(x, x_0) f(x_0)\, dx_0 + \int_{x+\varepsilon}^1 g(x, x_0) f(x_0)\, dx_0$$

where the integration has been split over two separate intervals for the purpose of treating the discontinuity in g_x. Using the rules for differentiating an integral (Leibnitz's rule) applied to this expression yields

$$\begin{aligned} u_x(x) &= \int_0^{x-\varepsilon} g_x(x, x_0) f(x_0)\, dx_0 + g(x, x-\varepsilon) f(x-\varepsilon) \\ &\quad + \int_{x+\varepsilon}^1 g_x(x, x_0) f(x_0)\, dx_0 - g(x, x+\varepsilon) f(x+\varepsilon) \\ &= \int_0^{x-\varepsilon} g_x(x, x_0) f(x_0)\, dx_0 + \int_{x+\varepsilon}^1 g_x(x, x_0) f(x_0)\, dx_0 \end{aligned}$$

Taking the derivative of this expression for u_x yields

$$\begin{aligned} u_{xx}(x) &= \int_0^{x-\varepsilon} g_{xx}(x, x_0) f(x_0)\, dx_0 + g_x(x, x-\varepsilon) f(x-\varepsilon) \\ &\quad + \int_{x+\varepsilon}^1 g_{xx}(x, x_0) f(x_0)\, dx_0 - g_x(x, x+\varepsilon) f(x+\varepsilon) \end{aligned}$$

The discontinuity in the first derivative yields

$$g_x(x, x-\varepsilon) f(x-\varepsilon) - g_x(x, x+\varepsilon) f(x+\varepsilon) = \frac{f(x)}{T}$$

Hence

$$u_{xx} = \int_0^{x-\varepsilon} g_{xx}(x, x_0) f(x_0)\, dx_0 + \int_{x+\varepsilon}^1 g_{xx}(x, x_0) f(x_0)\, dx_0 - \frac{f(x)}{T} \tag{10.39}$$

However, $g_{xx} = 0$ in the intervals specified in the two integrals. Thus, this last expression is just the equation $Lu = f$. The function $u(x)$ then becomes

$$u(x) = \int_0^1 g(x, y) f(y)\, \mathrm{d}y$$

which satisfies Equation (10.37) as well as the boundary condition.

Now note that $Gf = u$, so that G applied to $Lu = f$ yields $G(Lu) = Gf = u$, and hence $GLu = u$. Also, L applied to $Gf = u$ yields $LGf = Lu = f$, so that $LGf = f$. Thus, the operator G is clearly the inverse of the operator L.

In the same way, the Green's function can also be used to express the eigenvalue problem for the string. In fact,

$$\int_0^1 g(x, x_0)\theta(x_0)\, dx_0 = \mu\theta(x) \tag{10.40}$$

yields the eigenfunctions $\theta(x)$ for the operator L as defined in Equation (10.2), where $\mu = 1/\lambda$ and Equation (10.32) is defined by G.

To summarize, consider the slightly more general operator given by

$$Lw = a_0(x)w_{xx}(x) + a_1(x)w_x(x) + a_2(x)w(x) = 0 \tag{10.41}$$

with boundary conditions given by

$$B_1 w|_{x=0} = 0 \qquad \text{and} \qquad B_2 w|_{x=1} = 0 \tag{10.42}$$

Green's function for the operator given by Equations (10.41) and (10.42) is defined as the function $g(x, x_0)$ such that:

- $0 < x < 1, 0 < x_0 < 1$;
- $g(x, x_0)$ is continuous for any fixed value of x_0 and satisfies the boundary conditions in Equation (10.42);
- $g_x(x, x_0)$ is continuous except at $x = x_0$;
- as a function of x, $Lg = 0$ everywhere except at $x = x_0$;
- the jump discontinuity $g_x(x, x_0 + \varepsilon) - g_x(x, x_0 - \varepsilon) = 1/a_0(x)$ holds.

Green's function defines the inverse of the operators (10.41) and (10.42). Furthermore, the eigenvalue problem associated with the vibration problem can be recast as an integral equation as given by Equation (10.40). The Green's function approach can be extended to other operators. Both of these concepts are capitalized upon in the following chapters.

CHAPTER NOTES

The majority of this chapter is common material found in a variety of texts, only some of which are mentioned here. Section 10.2 introduces eigenfunctions and makes the connection between eigenfunctions and separation of variables as methods of solving boundary value problems arising in vibration analysis. The method of separation of variables is discussed in most texts on vibration as well as those on differential equations, such as the text by Boyce and DiPrima (2000). Eigenfunctions are also discussed in texts on operator theory, such as Naylor and Sell (1982). Few texts make an explicit connection between the two methods. The procedure is placed on a firm mathematical base in Chapter 11.

Section 10.3 examines, in an informal way, the method of modal analysis, a procedure made popular by the excellent texts by Meirovitch (1967, 2001). Here, however, the method is more directly related to eigenfunction expansions. Section 10.4 introduces damping as a simple velocity-proportional operator commonly used as a first attempt, as described in Section 9.7. Damping models represent a discipline by themselves. Here, a model of mathematical convenience is used. A brief look at using transform methods is provided in Section 10.5 for the sake of completeness. Transform methods have been developed by Yang (1992). Most transform methods are explained in detail in basic text, for instance by Churchill (1972). The last section on Green's functions follows closely the development in Stakgold (1979, 2000). Green's function methods provide a strong basis for the theory to follow in the next chapter. Most texts on applied mathematics in engineering discuss Green's functions.

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PROBLEMS

10.1 Compute the eigenvalues and eigenfunctions for the operator

$$L = -\frac{d^2}{dx^2}$$

with boundary conditions $u(0) = 0$, $u_x(1) + u(1) = 0$.

10.2 Normalize the eigenfunctions of problem 10.1.

10.3 Show that $u_{nm}(x, y) = A_{nm} \sin n\pi x \sin m\pi y$ is an eigenfunction of the operator

$$L = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

with boundary conditions $u(x, 0) = u(x, 1) = u(1, y) = u(0, y) = 0$. This is the membrane operator for a unit square.

10.4 Normalize the eigenfunctions of a membrane (clamped) of problem 10.3 and show that they are orthonormal.

10.5 Calculate the temporal coefficients, $a_n(t)$, for the problem of example 10.3.1.

10.6 Calculate the initial conditions required in order for $a_n(t)$ to have the form given in example 10.4.1.

10.7 Rewrite Equation (10.16) for the case where the eigenfunctions of L_1 are not the same as those of L_2.

10.8 Solve for the free longitudinal vibrations of a clamped-free bar in the special case where the damping is approximated by the operator $L_1 = 0.1I$, $EA = \rho = 1$, and the initial conditions are $w_t(x, 0) = 0$ and $w(x, 0) = 10^{-2}$.

10.9 Calculate Green's function for the operator given by $L = 10^6 \partial^2/\partial x^2$, $u(0) = 0$, $u_x(\ell) = 0$. This corresponds to a clamped bar.

10.10 Calculate Green's function for the operator $L = \partial^4/\partial x^4$, with boundary conditions $u(0) = u(1) = u_{xx}(1) = 0$. (*Hint*: The jump condition is $g_{xxx}(x, x + \varepsilon) - g_{xxx}(x, x - \varepsilon) = 1$.)

10.11 Normalize the eigenfunctions of example 9.31 and discuss the orthogonality conditions.