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11
Operators and the Free Response

11.1 INTRODUCTION

Just as knowledge of linear algebra and matrix theory is helpful in the study of lumped-parameter vibration problems, so a working knowledge of functional analysis and operator theory is useful in the study of the vibrations of distributed-parameter systems. A complete introduction to these topics requires a sound background in mathematics and is not possible within the limited space here. However, this chapter introduces some of the topics of relevance in vibration analysis. One of the main concerns of this section is to consider the convergence of the series expansions of eigenfunctions used in the separation of variables and modal analysis methods introduced in the previous chapters. The intent of this chapter is similar to that of Chapter 3, which introduced linear algebra as needed for discussing the free response of lumped-mass systems (also called finite-dimensional systems). The goal of this chapter is to provide a mathematical analysis of the methods used in Chapter 10.

In addition, some results are presented that examine the qualitative nature of the solution of linear vibration problems. In many instances, the describing differential equations cannot be solved in closed form. In these situations, knowing the nature of the solution rather than its details may be satisfactory. For example, knowing that the first natural frequency of a structure is bounded away from a driving frequency, rather than knowing the exact numerical value of the natural frequency, may be sufficient. Also, knowing whether a given structure will oscillate or not without computing the solution is useful. All this qualitative behavior is discussed in this chapter. As indicated in Chapter 6, qualitative results are very useful in design situations.

11.2 HILBERT SPACES

The definition of integration familiar to most engineers from introductory calculus is *Riemann integration*. Riemann integration can be defined on the interval $[a, b]$ as

$$\int_a^b f(x)\,dx = \lim_{\substack{n\to\infty \\ \Delta x\to 0}} \sum_{i=1}^{n} f(x_i)\,\Delta x_i \tag{11.1}$$

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where Δx_i is a small interval of the segment $b-a$, for instance, $(b-a)/n$. In most engineering applications, this type of integration is adequate. However, quite often a sequence of functions $\{f_n(x)\}$, defined for values of x in the interval $[a, b]$, converges to a function $f(x)$, i.e.,

$$\{f_n(x)\} \rightarrow f(x) \tag{11.2}$$

as n approaches infinity. Then, being able to conclude that

$$\lim_{n\rightarrow\infty}\left(\int_a^b f_n(x)\,dx\right) \rightarrow \int_a^b f(x)\,dx \tag{11.3}$$

is important. For example, when using modal analysis, this property is required. Unfortunately, Equation (11.3) is not always true for Riemann integration. The *Lebesgue integral* was developed to force Equation (11.3) to be true.

Lebesgue integration was developed using the concept of measurable sets, which is beyond the scope of this text. Thus, rather than defining Lebesgue integration directly, the properties are listed below:

1. If $f(x)$ is Riemann integrable, then it is Lebesgue integrable, and the two integrations yield the same value (the reverse is *not* true).
2. If α is a constant and $f(x)$ and $g(x)$ are Lebesgue integrable, then $\alpha f(x)$ and $g(x) + f(x)$ are Lebesgue integrable.
3. If $f^2(x)$ is Lebesgue integrable, then $\int f^2(x)\,dx = 0$ if and only if $f(x) = 0$ *almost everywhere* (i.e., everywhere except at a few points).
4. If $f(x) = g(x)$ almost everywhere, then $\int f^2(x)\,dx = \int g^2(x)\,dx$.
5. If $f_n(x)$ is a sequence of functions that are Lebesgue integrable over the interval $[a, b]$, if the sequence $\{f_n(x)\}$ converges to $f(x)$, and if for sufficiently large n there exists a function $F(x)$ that is Lebesgue integrable and $|f_n(x)| < F(x)$, then
   (a) $f(x)$ is Lebesgue integrable;
   (b) $\lim_{n\rightarrow\infty}\left\{\int_a^b f_n(x)\,dx\right\} = \int_a^b f(x)\,dx$

Any function $f(\mathbf{x}), \mathbf{x} \in \Omega$, such that$f^2(\mathbf{x})$is Lebesgue integrable, is called square integrable in the Lebesgue sense, denoted by $f \in \mathcal{L}_2^{\mathrm{R}}(\Omega)$, and, as will be illustrated, defines an important class of functions. In fact, the set of functions that are $\mathcal{L}_2^{\mathrm{R}}(\Omega)$ make up a linear space (see Appendix B or Naylor and Sell, 1982). In this notation, the subscript 2 denotes square integrable, the superscript R denotes the functions are all real, and Ω denotes the domain of integration.

Another important concept is that of linearly independent sets of functions. An arbitrary set of functions is said to be *linearly independent* if every finite subset of elements is linearly independent. Note that this definition is consistent with the notion of linear independence introduced in Chapter 3 and, in fact, is based on that definition.

The concept of a linear space, or vector space, is sufficient for the study of matrices and vectors. However, more mathematical structure is required to discuss operators. In particular, a linear space is defined to be an *inner product* space if, with every pair of elements u and

v in the space, there exists a unique complex number (u, v), called the inner product of u and v, such that:

- $(u, v) = (v, u)^*$, where the asterisk indicates the complex conjugate;
- $(\alpha u_1 + \beta u_2, v) = \alpha(u_1, v) + \beta(u_2, v)$;
- $(u, u) > 0$;
- $(u, u) = 0$ if and only if $u = 0$.

The inner product most often used in vibrations is that defined by the Lebesgue integral given in the form

$$(u, v) = \int_{\Omega} u(\mathbf{x}) v^*(\mathbf{x})\, d\Omega \tag{11.4}$$

Next, a few more definitions and results are required in order to mimic the structure used in linear algebra to analyze vibration problems. The *norm* of an element in a linear space is denoted by $||u||$ and is defined, in the case of interest here, by

$$||u(\mathbf{x})|| = \sqrt{(u(\mathbf{x}), u(\mathbf{x}))} \tag{11.5}$$

Note that $||u(\mathbf{x})|| = 0$ if and only if $u(\mathbf{x}) = 0$ almost everywhere. In addition, the norm satisfies the following conditions:

- $||u|| > 0$;
- $||\alpha u|| < |\alpha|\ ||u||$, where α is a scalar;
- $||u + v|| \leq ||u|| + ||v||$ with equality if and only if $v = \alpha u > 0$ (referred to as the *triangle inequality*).

This last set of conditions introduces even more structure on a linear space. A linear space with such a norm is called a *normed linear space*. The set $\mathcal{L}_2^{\mathrm{R}}(\Omega)$ can be shown to form an inner product space with the preceding definition of an inner product and a normed linear space with the preceding definition of a norm.

Note that, in the case of vectors, the scalar product $\mathbf{x}^{\mathrm{T}}\mathbf{x}$ satisfies the preceding definition of inner product. An important property of sets of elements is that of orthogonality. Just as in the case of vectors, two elements, u and v, of an inner product space are said to be *orthogonal* if $(u, v) = 0$. Orthogonality is used extensively in Section 10.3 with respect to modal analysis.

Based on the definitions of convergence used in calculus for scalars, a definition of convergence for elements of a normed linear space can be stated. Let $\{u_n(x)\}$ be a set of elements in a normed linear space, denoted by V. The sequence $\{u_n(x)\}$ *converges* to the element $u(x)$ in V if, for all $\varepsilon > 0$, there exists a number $N(\varepsilon)$ such that

$$\|u_n(x) - u(x)\| < \varepsilon$$

whenever $n > N(\varepsilon)$. This form of convergence is denoted by any of the following:

- $u_n \to u \qquad \text{as } n \to \infty;$
- $\lim_{n\to\infty} u_n(x) = u(x)$
- $\lim_{n\to\infty} \|u_n(x) - u(x)\| = 0$

In particular, the notation

$$u(x) = \sum_{n=1}^{\infty} u_n(x)$$

implies that the sequence of partial sums converges to $u(x)$, i.e.,

$$\lim_{k\to\infty} \sum_{n=1}^{k} u_n(x) = u(x)$$

Convergence defined this way is referred to as *strong convergence*, or *norm convergence*. As pointed out briefly in Section 9.3, this convergence is required when writing the modal expansion of Equation (10.11).

In the case of vectors, writing a series of weighted sums of eigenvectors is sufficient. The resulting sum is always finite, since the sum in the modal expansion contains a finite number of terms. However, since the sums in general normed linear spaces may have an infinite number of terms, convergence to some finite-valued function is not obvious.

As an aid in considering the convergence of a sequence of functions, a Cauchy sequence is used. A sequence $\{u_n(x)\}$ is defined to be a *Cauchy sequence* if, for all numbers $\varepsilon > 0$, there exists a number $N(\varepsilon)$ such that $||u_n(x) - u_m(x)|| < \varepsilon$ for every index m and n larger than $N(\varepsilon)$. An immediate consequence of this definition, and the triangle inequality, is that every convergent sequence is a Cauchy sequence. However, in general, not every Cauchy sequence converges. Requiring Cauchy sequences to converge leads to the concept of completeness. A normed linear space is defined as *complete* if every Cauchy sequence in that space converges in that space.

Note that, in a complete normed linear space, convergent sequences and Cauchy sequences are identical. The concept of completeness means that the limits of all of the sequences in the space that converge are also in that space. Comparing the set of real numbers with the set of rational numbers is analogous to the concept of completeness. The set of rational numbers is not complete, since one can construct a sequence of rational numbers that converges to an irrational number (such as the square root of two), which is not rational and hence is not in the set.

A complete normed linear space is called a *Banach space*. The set of all vectors of dimension n with real elements and with the norm defined by $||\mathbf{x}||^2 = \mathbf{x}^{\mathrm{T}}\mathbf{x}$ is a familiar example of a Banach space. The major difference in working with lumped-parameter vibration problems versus working with distributed-parameter vibration problems is based on the fact that every finite-dimensional normed linear space is complete and many common infinite-dimensional spaces are not. This possibility requires some concern over issues of convergence when using mode summation methods.

Example 11.2.1

Show by example that the space defined by the set of all continuous functions defined on the interval $[-1, 1]$ with norm

$$||u||^2 = \int_{-1}^{1} |u|^2\, dx$$

is not complete and also that a Cauchy sequence in the space does not converge to something in the space. Consider the sequence of continuous functions $u_n(x)$, where

$$u_n(x) = \begin{cases} 0, & -1 \leq x \leq 0 \\ nx, & 0 < x \leq \dfrac{1}{n} \\ 1, & \dfrac{1}{n} < x \leq 1 \end{cases}$$

A quick computation verifies that the sequence is a Cauchy sequence, but the sequence $\{u_n\}$ converges to the function $u(x)$ given by

$$u(x) = \begin{cases} 0, & -1 \leq x \leq 0 \\ 1, & 0 < x \leq 1 \end{cases}$$

which is *discontinuous* and hence is not an element in the space defined on the set of continuous functions.

In this last example, note that both $u_n(x)$ and $u(x)$ are square integrable in the Lebesgue sense. In other words, if, instead of requiring the linear space in the example to be the set of continuous functions, the set of square integrable functions in the Lebesgue sense is used, the space *is* a complete normed linear space. Since using modal analysis is desirable, mimicking the procedure used in finite dimensions, the natural choice of linear spaces to work in is $\mathcal{L}_2^{\mathrm{R}}(\Omega)$ or, when appropriate, $\mathcal{L}_2^{\mathrm{C}}(\Omega)$

Again, motivated by the method of modal analysis, it is desirable to equip the linear space with an inner product. Gathering all this mathematical structure together yields the class of functions most useful in vibration analysis. This space is called a *Hilbert space*, denoted by H. A Hilbert space is defined as a complete inner product space. Again, the set of real vectors with inner product $\mathbf{x}^T\mathbf{x}$ is an example of a Hilbert space. The Hilbert spaces of interest here are $\mathcal{L}_2^{\mathrm{R}}(\Omega)$ and $\mathcal{L}_2^{\mathrm{C}}(\Omega)$.

A further requirement placed on Hilbert spaces used in vibration applications is the assumption that the space is separable. A *separable Hilbert space* is a Hilbert space that contains a countable set of elements $\{f_n(x)\}$ such that, for any element f in the space and any positive real number ε, there exists an index N and a set of constants $\{\alpha_i\}$ such that

$$\left\| f(x) - \sum_{n=1}^{N} \alpha_n f_n(x) \right\| < \varepsilon \tag{11.6}$$

Here, the set $\{f_n\}$ is called a *spanning set* for the Hilbert space. These concepts and their significance are discussed in the next section. However, note again that finite-dimensional vector spaces are separable and that this property is useful in writing modal expansions. Both spaces $\mathcal{L}_2^{\mathrm{R}}(\Omega)$ and $\mathcal{L}_2^{\mathrm{C}}(\Omega)$ are separable Hilbert spaces. The symbol $\mathcal{H}$ is used to denote a general separable Hilbert space.

11.3 EXPANSION THEOREMS

In this section, the expansion theorem, or modal expansion, used informally in the last chapter is placed in a more rigorous setting by generalizing the Fourier series. Let $\{\phi_k\}$ be an infinite set of orthonormal functions in $\mathcal{H}$, i.e., $(\phi_k, \phi_i) = \delta_{ki}$. For any element $u \in \mathcal{H}$, the scalar (u, ϕ_k) is called the *Fourier coefficient* of u relative to the set $\{\phi_k\}$. Furthermore, the sum

$$\sum_{k=1}^{\infty}(u, \phi_k)\phi_k(x)$$

is called the *Fourier series* of $u(x)$ relative to the set $\{\phi_k\}$.

The following results (stated without proof) are useful in extending the Fourier theorem and in understanding how to use modal expansions in applications. Let $\{\phi_n\}$ again be an orthonormal set of elements in $\mathcal{H}$, let u also be an element in $\mathcal{H}$, and let $\{\lambda_k\}$ denote a set of complex scalars. Then, the following relationships hold

$$\left\| u - \sum_{k=1}^{n} \lambda_k \phi_k \right\| \geq \left\| u - \sum_{k=1}^{n} (u, \phi_k)\phi_k \right\| \tag{11.7}$$

$$\sum_{k=1}^{\infty} |(u, \phi_k)|^2 < \|u(x)\|^2 \quad \text{(called Bessel's inequality)} \tag{11.8}$$

and

$$\lim_{k\to\infty}(u, \phi_k) = 0 \tag{11.9}$$

This last result is referred to as the *Riemann–Lebesgue lemma*.

Furthermore, a famous result, the *Riesz–Fischer theorem*, relates the convergence of a modal expansion to the convergence of the coefficients in that expansion:

1. If $\sum_{k=1}^{\infty} |\lambda_k|^2 < \infty$, then $\sum_{k=1}^{\infty} \lambda_k\phi_k(x)$ converges to an element, u, in $\mathcal{H}$ and $\lambda_k = (u, \phi_k)$, i.e., the λ_k are the Fourier coefficients of the function $u(x)$.
2. If $\sum_{k=1}^{\infty} |\lambda_k|^2$ diverges, then so does the expansion $\sum_{k=1}^{\infty} \lambda_k\phi_k$.

A set of orthonormal functions satisfying (1) is called a *complete set* of functions. Recall that a complete space is a space in which Cauchy sequences converge. A complete set, on the other hand, is a set of elements in a space such that every element in the space can be represented as a series expansion of elements of the set. In general, an orthonormal set of functions $\{\phi_k\}$ in $\mathcal{H}$ is complete if any of the following relationships hold:

$$u = \sum_{k=1}^{\infty}(u, \phi_k)\phi_k \qquad \text{for each in } \mathcal{H} \tag{11.10}$$

$$||u||^2 = \sum_{k=1}^{\infty}|(u, \phi_k)|^2 \qquad \text{for each continuous } u \text{ in } \mathcal{H} \tag{11.11}$$

$$\text{If } (u, \phi_n) = 0 \text{ for each index } n, \text{ then } u = 0 \tag{11.12}$$

The second condition [Equation (11.11)] is referred to as *Parseval's equality*.

The preceding theorems generalize the concept of a Fourier series expansion of a function and provide a framework for modal expansions.

11.4 LINEAR OPERATORS

The idea of a linear operator was introduced in Chapter 10 as related to vibration problems. Here, the definition of a linear operator is formalized, and some properties of a linear operator are developed. The eigenfunctions and eigenvalues of operators are also discussed in detail, and the concept of adjoint is introduced.

A *subspace* of a linear space is a subset of elements of that space that again has the structure of a linear space. A linear operator is briefly defined in Section 10.2 as a mapping from one set of functions to another. Then the subspace $D(L)$ of the space $\mathcal{H}$ denotes the *domain* of the operator L and is the space of elements in $\mathcal{H}$ that the operator L is defined to act upon. A rule L is defined to be an operator if, for each $u \in D(L)$, there is a uniquely determined element Lu that lies in $\mathcal{H}$. An operator L is linear if, for every complex scalar α and β as well as for u and v in $D(L)$, the following is true:

$$L(\alpha u + \beta v) = \alpha L u + \beta L v \tag{11.13}$$

The operator L defines two other spaces of interest. The first is the *range space*, or *range*, of the operator L, denoted by $R(L)$; it is defined as the set of all functions $\{Lu\}$, where $u \in D(L)$. The domain and range of an operator are exactly analogous to the domain and range of a function. Another very important space associated with an operator is the *null space*. The null space of an operator L is the set of all functions $u \in D(L)$ such $Lu = 0$. This space is denoted by $N(L)$ and corresponds to rigid body modes in a structure. The spaces $R(L)$ and $N(L)$ are in fact subspaces of $\mathcal{H}$.

Obvious examples of these various spaces associated with an operator are the transformation matrices of Chapter 3. In fact, linear operator theory is an attempt to generalize the theory of matrices and linear algebra.

An important difference between matrices and operators can be pointed out by defining *equality* of two operators. Two operators L_1 and L_2 are equal if and only if $D(L_1) = D(L_2)$ and $L_1 u = L_2 u$ for all $u(x) \in D(L_1)$.

Example 11.4.1

Consider the linear operator associated with the string equation with fixed ends. Define this operator's domain and null space. The operator is

$$L = -\frac{\partial^2}{\partial x^2}$$

The domain, $D(L)$, consists of all functions in $\mathcal{L}_2^{\mathrm{R}}(0, \ell)$ having two derivatives and satisfying the boundary conditions $u(0) = u(\ell) = 0$. The null space of L is the set of all functions u in $D(L)$ such that $u'' = 0$. Integrating $u'' = 0$ requires that $u = ax + b$, where a and b are constants of integration. However, elements in $D(L)$ must also satisfy $a(0) + b = 0$ and $a\ell + b = 0$, so that $a = b = 0$. Thus, the null space of this operator consists of only the zero function.

An operator L is called *one-to-one* if $Lu_1 = Lu_2$ holds if and only if $u_1 = u_2$. A linear operator L has an *inverse*, denoted by L^{-1} and defined by $L^{-1}u = v$, if and only if $u = Lv$. Operator L is said to be nonsingular in this case and singular if L^{-1} does not exist. Note that for nonsingular operators

$$
\begin{aligned}
&D(L^{-1}) = R(L) \\
&R(L^{-1}) = D(L) \\
&LL^{-1}u = u \qquad \text{for all } u \text{ in } D(L^{-1}) = R(L)
\end{aligned}
$$

and

$$L^{-1}Lu = u \qquad \text{for all } u \text{ in } D(L) = R(L^{-1})$$

Also, L^{-1} is one-to-one. The inverse operator, as shown in Section 10.6, of a differential operator often turns out to be an integral operator defined by a Green's function.

The concept of null space and inverse are also closely related to the eigenvalues of an operator. In particular, suppose zero is an eigenvalue of L_1. Then $L_1 u = 0$ for $u \neq 0$ (since eigenfunctions are never zero). Thus, there exists a nonzero function u in the null space of the operator L. Actually, a stronger statement holds true. If L is a one-to-one operator, L has an inverse, and $N(L)$ contains only zero if and only if zero is not an eigenvalue of the operator L.

An operator is bounded if there exists a finite constant $c > 0$ such that

$$\|Lu\| < c\,\|u\|$$

for each function $u(x)$ in $D(L)$. A related operator property is continuity. An operator L is *continuous* at $u \in D(L)$ if, whenever $\{u_n\}$ is a sequence of functions in $D(L)$ with limit u, $Lu_n \to Lu$. This definition of continuity is often abbreviated as $u_n \to u \Rightarrow Lu_n \to Lu$. For linear operators, this definition is equivalent to requiring L to be continuous at every element in $D(L)$. In addition, a linear operator is continuous if and only if it is a bounded operator.

Example 11.4.2

Differential operators are not bounded. Consider $L = d/dx$ with $D(L)$ consisting of all functions in $\mathcal{L}_2^{\mathrm{R}}(0, 1)$ such that Lu is in $\mathcal{L}_2^{\mathrm{R}}(0, 1)$. This operator is not bounded since the element $u(x) = \sin(n\pi x) \in \mathcal{L}_2^{\mathrm{R}}(0, 1)$. Then $||u|| = \text{constant} = a$, independent of n, and

$$||Lu|| = n\left(\int \cos^2 n\pi x\, dx\right)^{1/2} = nb$$

where b is a constant. Hence, there is a function $u \in D(L)$ such that $||Lu|| > ||u||$, and the operator cannot satisfy the definition of a bounded operator because n may be arbitrarily large.

Example 11.4.3

Integral operators are bounded. Let $u \in \mathcal{L}_2^{\mathrm{R}}(0, 1)$ and define the operator L by

$$Lu(x) = \int_0^x u(s)\,ds = f(x)$$

Next, consider $f(x)$ as defined earlier and square its modulus to obtain

$$|f(x)|^2 = \left|\int_0^x u(s)\,ds\right|^2$$

However, the Cauchy–Schwarz inequality yields

$$\left|\int_0^x u(s)\,ds\right|^2 < \int_0^1 |1|^2 ds \int_0^1 |u(s)|^2\,ds = \|u\|$$

Hence, $||Lu|| < ||u||$, and this operator is bounded.

The operators used to describe the linear vibrations of common distributed-parameter structures are unbounded. However, the differential operators for strings, beams, membranes, and so on, all have inverses defined by Green's functions. Green's functions define integral operators, which are bounded. This connection of vibration equations to a bounded operator is significant in verifying convergence and eigenfunction expansions.

Another important operator is called the *adjoint operator*, which is defined in the following paragraphs and is basically the generalization of the transpose of a real matrix. First consider a linear operator T, which maps elements in $\mathcal{H}$ into a set of complex numbers. Such an operator is called a *linear functional*. In particular, consider the linear functional defined by an inner product. Let the linear functional T_v be defined as

$$T_v u = (u, v) = \int_\Omega uv^*\,d\Omega \tag{11.14}$$

where $u \in \mathcal{H}$, and v is a fixed element in $\mathcal{H}$. The action of T_v defines a bounded operator.

The following result, called the *Riesz representation theorem*, allows the adjoint operator for a bounded operator to be defined. Let T be a bounded linear functional defined on $\mathcal{H}$. Then there exists a unique element, f, in $\mathcal{H}$ such that

$$Tu = (u, f) \tag{11.15}$$

The significance of the Riesz representation theorem is as follows. Suppose that the operator L is bounded, and u and v are in $\mathcal{H}$. Then

$$Tu = (Lu, v) \tag{11.16}$$

defines a bounded linear functional. However, via the Riesz representation theorem, there exists a unique element f in $\mathcal{H}$ such that

$$Tu = (u, f) \tag{11.17}$$

Hence, Equation (11.16) yields

$$(Lu, v) = (u, f) \tag{11.18}$$

where $f \in \mathcal{H}$ is unique. The unique element f is denoted as $f = L^*v$, where L^* is defined as the *adjoint* of the operator L. Then, Equation (11.18) becomes

$$(Lu, v) = (u, L^*v) \tag{11.19}$$

The adjoint defined this way is linear and unique. The adjoint operator L^* is also bounded if L is bounded.

Unfortunately, the Riesz representation theorem and the preceding discussion of adjoints only hold for bounded operators. The equations of interest in vibrations, however, yield unbounded operators. Since the inverses of these unbounded operators are in fact bounded, the idea of an adjoint operator for unbounded operators is still used. However, to denote that a formal proof of existence of the adjoint operator for unbounded operators does not exist, the adjoint of an unbounded operator is often referred to as a *formal adjoint*.

Let L be an unbounded differential operator with domain $D(L)$. The operator L^* defined on a domain $D(L^*)$ is the formal adjoint of L if

$$(Lu, v) = (u, L^*v), \qquad u \in D(L), \qquad v \in D(L^*) \tag{11.20}$$

Note that $D(L)$ is characterized by certain boundary conditions. The domain $D(L^*)$ will also be characterized by possibly different boundary conditions called *adjoint boundary conditions*.

An operator (possibly unbounded) is defined to be *formally self-adjoint* if

$$D(L) = D(L^*) \qquad \text{and} \qquad Lu = L^*u \qquad \text{for } u \in D(L) \tag{11.21}$$

The prefix *formal* is not always used, however. It is important to note that, if L is formally self-adjoint, then, for all $u, v \in D(L)$,

$$(Lu, v) = (u, Lv) \tag{11.22}$$

which follows from Equation (11.21). This last relationship is used to define a related class of operators. An operator L is *symmetric* if Equation (11.22) holds for all pairs u and v in $D(L)$.

Note that the difference between the definitions of symmetric and formally self-adjoint is the domain requirement of Equation (11.21). That is, if the form of the operator and its adjoint are the same, then the operator is symmetric. If, in addition, both the operator and the adjoint operator have the same boundary conditions, then the operator is self-adjoint. If L is a bounded operator, then $D(L) = \mathcal{H}$, and formally self-adjoint and symmetric mean the same. If L is unbounded, however, an operator may be symmetric but not self-adjoint [i.e., $D(L) \neq D(L^*)$]. A formally self-adjoint operator, however, is a symmetric operator.

Example 11.4.4

Consider the operator for a fixed–fixed string and determine its adjoint. Here, L is of the form

$$L = -\alpha \frac{\partial^2}{\partial x^2}$$

and its domain is defined as

$$D(L) = \left\{ u | u(0) = u(\ell) = 0, u, u', u'' \in \mathcal{L}_2^{\mathrm{R}}(0, \ell) \right\}$$

The right-hand side of this expression denotes the set of all functions $u(x)$ such that the boundary conditions are satisfied (i.e., $u(0) = u(\ell) = 0$) and the functions u, u', and u'' belong in the set $\mathcal{L}_2^{\mathrm{R}}(0, \ell)$. Calculating the linear product (Lu, v) for $u \in D(L)$ and $v \in D(L^*)$ yields

$$(Lu, v) = -\alpha \int_0^\ell u'' v \, dx$$

Integrating by parts twice yields

$$(Lu, v) = [-\alpha u'(\ell)v(\ell) + \alpha u'(0)v(0) + \alpha u(\ell)v'(\ell) - \alpha u(0)v'(0)] - \alpha \int_0^\ell u v'' \, dx$$

Since $u \in D(L)$, the above reduces to

$$(Lu, v) = [-\alpha u'(\ell)v(\ell) + \alpha u'(0)v(0)] + (u, Lv) \tag{11.23}$$

Now, if both u and v are in $D(L)$, then $v(\ell) = v(0) = 0$ and Equation (11.23) shows that L is a symmetric operator. To see that L is in fact formally self-adjoint, however, consider Equation (11.23) with $v \in D(L^*)$. Since $v \in D(L^*)$, the integration by parts requires v, v', and v'' to be in $\mathcal{L}_2^{\mathrm{R}}(0, \ell)$. In order for $(Lu, v) = (u, Lv)$ to be satisfied, the term in brackets in Equation (11.23) must be zero, i.e.,

$$-\alpha u'(\ell)v(\ell) + \alpha u'(0)v(0) = 0$$

However, since $u'(\ell)$ and $u'(0)$ are arbitrary [i.e., there is no restriction on these values in $D(L)$], $v(\ell)$ and $v(0)$ must both be zero. Thus

$$D(L^*) = \left\{ v | v(0) = v(\ell) = 0, v, v', v'' \in \mathcal{L}_2^{\mathrm{R}}(0, \ell) \right\} = D(L)$$

and this operator is in fact formally self-adjoint. The boundary conditions $v(0) = v(\ell) = 0$ are the adjoint boundary conditions.

In order for a differential operator to be symmetric, it must be of even order (problem 11.20). Most of the physical structures examined so far are self-adjoint with respect to most boundary conditions. The following example illustrates a symmetric, nonself-adjoint operator.

Example 11.4.5

Consider the operator $L = \partial^2/\partial x^2$ on the domain

$$D(L) = \{u | u(0) = u'(0) = 0,\ u, u', u'' \in \mathcal{L}_2^{\mathrm{R}}(0, \ell)\}$$

Equation (11.23) still holds with $a = -1$. However, the term in brackets is now

$$[u'(\ell)v(\ell) - u(\ell)v'(\ell)] = 0$$

Since $u'(\ell)$ and $u(\ell)$ are arbitrary, this last expression requires that $v(\ell) = v'(\ell) = 0$. Hence, $D(L^*) = \{v | v, v', v'' \in \mathcal{L}_2^{\mathrm{R}}(0, \ell) \text{ and } v(\ell) = v'(\ell) = 0\} \neq D(L)$ and the operator is not self-adjoint. However, this operator is symmetric.

Symmetric operators (and thus self-adjoint operators) have special properties, as do symmetric matrices, which are useful in the study of vibrations. If L is symmetric, the following are true:

1. (Lu, u) is real.
2. If L has eigenvalues, they are all real.
3. Eigenfunctions corresponding to distinct eigenvalues are orthogonal.

Furthermore, if L is self-adjoint, it is called *positive definite* if $(Lu, u) > 0$ for all nonzero $u \in D(L)$ and *positive semidefinite* if $(Lu, u) \geq 0$ for all nonzero $u \in D(L)$.

Example 11.4.6

Consider the operator of example 11.4.4 and show that it is positive definite. Start by calculating (Lu, u) for an arbitrary element $u \in D(L)$. Integration by parts yields

$$\begin{aligned}(Lu, u) &= -(u', u)\Big|_0^\ell + \int_0^\ell (u')^2\, dx \\ &= \int_0^\ell (u')^2\, dx \geq 0\end{aligned}$$

where the boundary conditions eliminate the constant terms. This calculation shows that the operator L is positive semidefinite. To see that the operator is, in fact, strictly positive definite, note that, if $u' = 0$, then $u = c$, a constant. However, $u(0) = u(\ell) = 0$ must be satisfied, so that u must be zero, contradicting the semidefinite condition. Thus, $(Lu, u) > 0$ and L is in fact positive definite.

In the case of matrices, a symmetric matrix is positive definite if and only if its eigenvalues are all positive real numbers. For an operator, a weaker version of this statement holds.

A positive definite operator has positive eigenvalues (assuming it has eigenvalues). With further assumptions, which are clarified in the next section, a stronger statement can be made that is more in line with the matrix result.

11.5 COMPACT OPERATORS

In this section, the last major mathematical requirement for eigenfunction expansions is considered. Compact operators are defined and some of their principal properties examined. Self-adjoint compact operators have essentially the same eigenstructure as symmetric matrices and hence are ideal for modal analysis.

The notion of a compact operator is based on restricting a bounded operator. Let $\{u_n\}$ be a uniformly bounded, infinite sequence in $\mathcal{H}$. A *uniformly bounded sequence* is one for which $\|u_n\| < M$ for all values of n (i.e., M does not depend on n). Let L be a bounded linear operator defined on $\mathcal{H}$. The operator L is defined to be *compact* if, from the sequence $\{Lu_n\}$, one can extract a subsequence, denoted by $\{(Lu_n)_k\}$, that is a Cauchy sequence. Another way to describe a compact operator is to note that a compact operator maps bounded sets into compact sets. A compact set is a set such that each sequence of elements in the set contains a convergent subsequence. In this way, the notion of a compact operator is related to a compact set.

The idea of a compact operator is stronger than that of a bounded operator. In fact, if an operator is compact, it is also bounded. However, bounded operators are not necessarily compact. Since bounded operators are continuous operators and compactness requires more of the operator, compact operators are also called *completely continuous* operators.

The identity operator is an example of a bounded operator, i.e., $\|Iu\| < a\|u\|$, which is not necessarily compact. Every bounded operator defined on a finite-dimensional Hilbert space is compact, however. Bounded operators are compact because every set containing a finite number of elements is compact.

Consider next the integral operator defined by Green's function. Such operators are compact. In fact, if $g(x, y)$ is any continuous function where x and y are both in the interval $(0, \ell)$, then the integral operator defined by

$$Lu = \int_0^\ell g(x, y)u(y)\, dy \tag{11.24}$$

is a compact operator on $\mathcal{L}_2^{\mathrm{R}}(0, \ell)$.

Self-adjoint compact operators have the desired expansion property – namely, if L is compact and self-adjoint on $\mathcal{H}$, then L can be shown to have nonzero eigenvalues $\{\mu_n\}$ and orthonormal eigenfunctions $\{\Theta_n\}$. Next, let $u(x)$ be any element in $\mathcal{H}$; $u(x)$ can then be represented as the generalized Fourier series

$$u(x) = \sum_{n=1}^{\infty}(u, \Theta_n)\Theta_n + u_0(x) \tag{11.25}$$

where $u_0(x)$ lies in the null space of L. Furthermore, the function Lu can also be represented as

$$Lu(x) = \sum_{n=1}^{\infty}\mu_n(u, \Theta_n)\Theta_n \tag{11.26}$$

Note that, if L is nonsingular, $u_0(x) = 0$. Also note that, from comparing Equation (11.10) with Equation (11.25), the eigenfunctions of a compact self-adjoint operator are complete in $\mathcal{H}$. These two powerful results form the backbone of modal analysis of distributed-parameter systems.

Expression (11.26) also allows a more concrete statement about the relationship between the eigenvalues of an operator and the definiteness of an operator. In fact, a compact self-adjoint operator L is positive definite if and only if each of its eigenvalues are positive real numbers.

11.6 THEORETICAL MODAL ANALYSIS

In this section, the idea of a compact operator, along with the associated expansion, is applied to a generic model of a differential equation describing the linear vibration of a distributed-parameter system. This theory results in the modal analysis of such structures and provides a firm mathematical foundation for the material presented in Sections 10.3 and 10.4.

Consider again Equations (10.1), repeated here for convenience:

$$\begin{aligned} w_{tt}(x,t) + L_2 w(x,t) &= 0, & x &\in \Omega, & t &> 0 \\ Bw &= 0, & x &\in \partial\Omega, & t &> 0 \\ w(x,0) &= w_0(x), & w_t(x,0) &= \dot{w}_0(x) & t &= 0 \end{aligned} \tag{11.27}$$

With the additional assumptions that the (nonsingular) operator L_2 is self-adjoint and has a compact inverse, the following shows that the sum of Equation (10.3), i.e., the modal expansion of the solution, converges.

Since L_2^{-1} is compact, the eigenfunctions of L_2^{-1} are complete. As noted before, these eigenfunctions are also those of L_2, so that the eigenfunctions of L_2 are complete. As a result, the solution $w(x,t)$, considered as a function of x defined on the Hilbert space $\mathcal{H}$ for a fixed value of t, can be written as

$$w(x,t) = \sum_{n=1}^{\infty} a_n(t)\Theta_n(x) \tag{11.28}$$

where it is anticipated that the Fourier coefficient a_n will depend on the fixed parameter t, $t > 0$. From Equation (11.10) the coefficient $a_n(t)$ is

$$a_n(t) = \int_{\Omega} w(x,t)\Theta_n \, d\Omega = (w, \Theta_n). \tag{11.29}$$

Multiplying Equations (11.27) by $\Theta_n(x)$ and integrating over Ω yields

$$\int_{\Omega} w_{tt}\Theta_n(x)\; d\Omega + \int_{\Omega} L_2 w\Theta_n(x)\; d\Omega = 0 \tag{11.30}$$

Equation (11.30) can be rewritten as

$$\frac{\partial^2}{\partial t^2}[(w, \Theta_n)] + (L_2 w, \Theta_n) = 0 \tag{11.31}$$

Using the self-adjoint property of L_2 and the fact that Θ_n is an eigenfunction of L_2 with corresponding eigenvalue λ_n yields

$$\frac{\partial^2}{\partial t^2}[(w, \Theta_n)] + \lambda_n(w, \Theta_n) = 0$$

or, from Equation (11.29),

$$\ddot{a}_n(t) + \lambda_n a_n(t) = 0, \qquad t > 0 \tag{11.32}$$

This expression, along with the appropriate initial conditions, can be used to calculate $a_n(t)$. In particular, if L_2 is positive definite, $\lambda_n > 0$ for all n, then

$$a_n(t) = \frac{\dot{a}_n(0)}{\omega_n}\sin\omega_n t + a_n(0)\cos\omega_n t \tag{11.33}$$

where

$$\begin{aligned} a_n(0) &= \int_\Omega w(x, 0)\Theta_n(x)\; d\Omega \\ \dot{a}_n(0) &= \int_\Omega w_t(x, 0)\Theta_n(x)\; d\Omega \\ \omega_n &= \sqrt{\lambda_n} \end{aligned}$$

These formulae are, of course, consistent with those developed formally in Section 10.3. Here, however, the convergence of Equation (11.28) is guaranteed by the compactness theorem of Section 11.5.

As indicated in Section 10.4, this procedure can be repeated for damped systems if the operators L_1 and L_2 commute on a common domain. This result was first pointed out formally by Caughey and O'Kelly (1965), but the development did not concern itself with convergence. The approach taken by Caughey and O'Kelley, as well as in Section 10.4, is to substitute the series of Equation (11.28) into Equation (10.14). Unfortunately, this substitution raises the issue of convergence of the derivative of the series. The convergence problem is circumvented by taking the approach described in Equations (11.30) through (11.32).

Repeating this procedure for the damped system, as described by Equation (10.14), requires that L_1 and L_2 have the same eigenfunctions and that L_2 is self-adjoint with compact inverse. Multiplying Equation (10.14) by $\Theta_n(x)$ and integrating yields

$$\frac{d^2}{dt^2}[(w, \Theta_n)] + \frac{d}{dt}[(L_1 w, \Theta_n)] + (L_2 w, \Theta_n) = 0 \tag{11.34}$$

Using the property that L_1 and L_2 are self-adjoint, and denoting the eigenvalues of L_1 by $\lambda_n^{(1)}$ and those of L_2 by $\lambda_n^{(2)}$, Equation (11.34) becomes

$$\ddot{a}_n(t) + \lambda_n^{(1)}\dot{a}_n(t) + \lambda_n^{(2)}a_n(t) = 0 \tag{11.35}$$

Equations (11.32) and (11.35) constitute a theoretical modal analysis of a distributed-mass system. Equation (11.34) is solved using the methods of Section 1.3 from initial conditions determined by using mode orthogonality.

11.7 EIGENVALUE ESTIMATES

As indicated previously, knowledge of the eigenvalues (natural frequencies) of a system provides knowledge of the dynamic response of a structure. Unfortunately, the eigenvalue problem for the operators associated with many structures cannot be solved. Having no solution requires the establishment of estimates and bounds on the eigenvalues of operators. Note that, while this section is an extension of Section 3.7 on eigenvalue estimates for matrices, the operator problem is more critical because in the matrix case estimates are primarily used as an analytical tool, whereas in the operator case the estimates are used where solutions do not even exist in closed form.

Consider first the conservative vibration problems of Equation (10.1) for the case where the operator L is self-adjoint and positive definite with compact inverse. As noted in Section 11.6, these assumptions guarantee the existence of a countable infinite set of positive real eigenvalues $\{\lambda_i\}$, which can be ordered as

$$0 < \lambda_1 < \lambda_2 < \cdots < \lambda_n < \cdots \tag{11.36}$$

with corresponding orthonormal eigenfunctions, $\{\Theta_n(x)\}$, complete in $\mathcal{L}_2^{\mathrm{R}}(\Omega)$.

Next, further assume that the operator L is *coercive*, i.e., that there exists a constant c such that

$$c\|u\|^2 < (Lu, u) \tag{11.37}$$

for all $u \in D(L)$. Thus

$$\frac{(Lu, u)}{\|u\|^2} > c \tag{11.38}$$

for all $u \in D(L)$. Thus, the quantity $(Lu, u)/\|u\|^2$ is bounded below, and therefore there is a *greatest lower bound*, denoted by glb, of this ratio. Define the functional $R(u)$ by

$$R(u) = \frac{(Lu, u)}{\|u\|^2} \tag{11.39}$$

The functional $R(u)$ is called the *Rayleigh quotient* of the operator L.

Since the eigenfunctions of L are complete, Equation (11.11) yields

$$\|u\|^2 = \sum_{n=1}^{\infty} |(u, \Theta_n)|^2$$

and

$$(Lu, u) = \sum_{n=1}^{\infty} \lambda_n |(u, \Theta_n)|^2 \tag{11.40}$$

Hence, the Rayleigh quotient can be written as

$$R(u) = \frac{\sum \lambda_n |(u, \Theta_n)|^2}{\sum |(u, \Theta_n)|^2} > \lambda_1 \tag{11.41}$$

since Equation (11.7) implies that

$$\sum \lambda_n |(u, \Theta_n)|^2 > \lambda_1 \sum |(u, \Theta_n)|^2 \tag{11.42}$$

Here, the summation limits have been suppressed. Also, note that $R(\Theta_1) = \lambda_1$, so that

$$\lambda_1 = \min(\lambda_i) = \underset{\substack{u \neq 0 \\ u \in D(L)}}{\text{glb}} [R(u)] \tag{11.43}$$

As in the matrix case, the Rayleigh quotient yields a method of finding an upper bound to the eigenvalues of a system.

Example 11.7.1

Consider the operator $L = -\partial^2/\partial x^2$ defined on the domain $D(L) = \{u | u(0) = u(1) = 0, u, u', u'' \in \mathcal{L}_2^{\mathrm{R}}(0, 1)\}$. Estimate the first natural frequency by using the Rayleigh quotient.

Note that the function $u(x) = x(1 - x)$ is in $D(L)$. Calculation of the Rayleigh quotient then yields

$$R(u) = \frac{-(u'', u)}{(u, u)} = \frac{1/3}{1/30} = 10$$

A calculation of the exact value of λ_1 for this operator yields

$$\lambda_1 = (\pi^2) < 10$$

so that the Rayleigh quotient in this case provides an upper bound to the lowest eigenvalue.

Bounds are also available for the other eigenvalues of an operator. In fact, with the previously mentioned assumptions on the operator L, the domain of L can be split into two subspaces M_k and $M_k^{\perp}$ (read 'M_k perp') defined by the eigenfunctions of L. Let $M_k = \{\Theta_1, \Theta_2, \ldots, \Theta_k\}$ be the set of the first k eigenfunctions of L, and $M_k^{\perp}$ be the set of remaining eigenfunctions, i.e., $M_k^{\perp} = \{\Theta_{k+1}, \Theta_{k+2}, \ldots\}$. From these considerations (see, for instance, Stakgold, 1967, 2000a) the following holds:

$$\lambda_k = \min_{u \in M_{k-1}^{\perp}} [R(u)] = \max_{u \in M_k} [R(u)] \tag{11.44}$$

This formulation of the eigenvalues of an operator as a minimum or a maximum over sets of eigenfunctions can be further extended to extremals over arbitrary subspaces. Equation (11.44) is called the *Courant minimax principle*. Again, with L satisfying the assumptions of this section, let E_k be any k-dimensional subspace of $D(L)$. Then

$$\lambda_k = \min_{E_k \in D(L)} \left[\max_{\substack{u \in E_k \\ u \neq 0}} (R(u)) \right] = \max_{E_k \in D(L)} \left[\max_{\substack{u \in E_{k-1}^{\perp} \\ u \neq 0}} (R(u)) \right] \tag{11.45}$$

where the value of $u(x)$ satisfying Equation (11.45) becomes Θ_k. The minimum over $E_k \in D(L)$ refers to the minimum over all the subspaces E_k of dimension k contained in

$D(L)$. The maximum value of $R(u)$, $u \in E_k$, refers to the maximum value of $R(u)$ for each element u in the set E_k. The difference between Equations (11.44) and (11.45) is that Equation (11.44) is restricted to subspaces generated by the eigenfunctions of L, whereas in Equation (11.45) the subspaces are any subspace of $D(L)$.

11.8 ENCLOSURE THEOREMS

The Rayleigh quotient and formulations of the eigenvalue problem of the previous section provide a means of estimating an upper bound of the eigenvalues of an operator by examining sets of arbitrary functions in the domain of the operator. In this section, lower bounds and enclosure bounds are examined. Furthermore, bounds in terms of related operators are examined. In this way, eigenvalues of operators that are difficult or impossible to calculate are estimated in terms of operators with known eigenvalues.

The first two results follow from the definition of the definiteness of an operator. This definition can be used to build a partial ordering of linear operators and to provide an eigenvalue estimate. For two self-adjoint operators L_1 and L_2, the operator inequality denoted by

$$L_1 \leq L_2 \tag{11.46}$$

is defined to mean that

$$\mathcal{H} \supset D(L_1) \supset D(L_2) \tag{11.47}$$

and

$$(L_1 u, u) \leq (L_2 u, u) \qquad \text{for all } u \in D(L_2) \tag{11.48}$$

where Equation (11.47) denotes that $D(L_2)$ is a subset of $D(L_1)$, and so on.

If Equation (11.48) holds with strict equality, i.e., if L_1 and L_2 have the same form, with L_1 defined on a subspace of L_2, and if L_1 and L_2 are positive definite with compact inverses, then

$$\lambda_i^{(1)} \leq \lambda_i^{(2)}, \qquad i = 1, 2, \ldots \tag{11.49}$$

Here, $\lambda_i^{(1)}$ denotes the eigenvalues of L_1, and $\lambda_i^{(2)}$ denotes those of the operator L_2. This inequality is called the first *monotonicity principle*.

Example 11.8.1

Consider the two operators defined by $L_1 = L_2 = -\partial^2/\partial x^2$ with domains $D(L_1) = \{u \mid u(0) = u(\ell_1) = 0, u, u', u'' \in \mathcal{L}_2^{\mathrm{R}}(0, \ell_1)\}$ and $D(L_2) = \{u \mid u(0) = u(\ell_2) = 0, u, u', u'' \in \mathcal{L}_2^{\mathrm{R}}(0, \ell_2)\}$. Consider the case with $\ell_1 \leq \ell_2$. Then redefine the domain $D(L_1)$ to be the completion of the set of functions that vanish outside $D(L_1)$, i.e., for $x > \ell_1$. Then, $D(L_2) \supset D(L_1)$ and expression (11.47) and inequality (11.49) are satisfied so that

$$L_2 < L_1$$

Thus, inequality (11.49) yields (note the indices are interchanged)

$$\lambda_i^{(2)} \le \lambda_i^{(1)}$$

To see that this is true, note that $\lambda_i^{(2)} = \sqrt{\mathrm{i}\pi/\ell_2} \le \sqrt{\mathrm{i}\pi/\ell_1} = \lambda_i^{(1)}$. This example shows that shrinking the domain of definition of the problem increases the eigenvalues, as expected.

The trick in using the first monotonicity principle is the ability to extend the domain $D(L_2)$ so that it can be considered as a subspace of $D(L_1)$. Extending the domain works in the example because the boundary condition is $u = 0$ along the boundary. The method fails, for instance, for the membrane equation with clamped boundaries and a hole removed (Weinberger, 1974).

The preceding example was chosen to illustrate that the principle works. The use of this principle is more interesting, however, in a situation where one of the boundaries is such that the eigenvalues cannot be calculated, i.e., as in the case of an odd-shaped membrane. In this case, the unknown eigenvalues, and hence natural frequencies, can be bracketed by two applications of the first monotonicity theorem.

For instance, if the eigenvalues of a membrane of irregular shape are required, the eigenvalues of inscribed and circumscribed rectangles can be used to provide both upper and lower bounds of the desired eigenvalues. Thus, the monotonicity principle can also be thought of as an *enclosure theorem* (see problem 11.17).

If the operators L_1 and L_2 are positive definite and of different form, i.e., if the equality in expression (11.48) does not hold, then inequality (11.49) is known as the *second monotonicity theorem*. The following example illustrates how the monotonicity results can be used to create enclosures for the eigenvalues of certain operators.

Example 11.8.2

Consider the membrane operator $L = -\nabla^2$, as defined by Equation (9.53). In particular, consider the three operators

$$L_1 = -\nabla^2, \qquad D(L_1) = \left\{ u \,\middle|\, \frac{\partial u}{\partial n} + k_1 u = 0 \quad \text{on } \partial\Omega,\ u, u_x, u_y, u_{xy}, u_{xx}, u_{yy} \in \mathcal{L}_2^{\mathrm{R}}(\Omega) \right\}$$

$$L_2 = -\nabla^2, \qquad D(L_2) = \left\{ u \,\middle|\, \frac{\partial u}{\partial n} + k_2 u = 0 \quad \text{on } \partial\Omega,\ u, u_x, u_y, u_{xy}, u_{xx}, u_{yy} \in \mathcal{L}_2^{\mathrm{R}}(\Omega) \right\}$$

$$L_3 = -\nabla^2, \qquad D(L_3) = \left\{ u | u = 0 \quad \text{on } \partial\Omega,\ u, u_x, u_y, u_{xy}, u_{xx}, u_{yy} \in \mathcal{L}_2^{\mathrm{R}}(\Omega) \right\}$$

where $k_1 > k_2 > 0$. Compare the eigenvalues of these three operators.

With $k_1 > k_2$, integration yields

$$(u, L_2 u) < (u, L_1 u)$$

For all $u \in D(L_2)$ and using the second monotonicity principle we have

$$\lambda_i^{(2)} \le \lambda_i^{(1)}$$

Comparing operators L_3 and L_1, note that $D(L_3) \supset D(L_1)$, so that application of the first monotonicity principle yields

$$\lambda_i^{(2)} \leq \lambda_i^{(1)} \leq \lambda_i^{(3)}$$

11.9 OSCILLATION THEORY

In this section, the damped system of Equation (10.14) with the assumption stated in Section 11.6 is considered. Under these assumptions, the solution of Equation (10.14) is expressible as the convergent series

$$w(x, t) = \sum_{n=1}^{\infty} a_n(t)\Theta_n(x) \tag{11.50}$$

where the coefficients $a_n(t)$ satisfy Equations (11.35), i.e.,

$$\ddot{a}_n(t) + \lambda_n^{(1)}\dot{a}_n(t) + \lambda_n^{(2)}a_n(t) = 0, \qquad n = 1,\ 2,\ 3, \ldots \tag{11.51}$$

The topic of interest in this section is the nature of the temporal solution $a_n(t)$ subjected to arbitrary initial conditions. This section is an extension of the oscillation results presented in Section 3.6 for lumped-mass systems to those distributed-mass structures described by Equation (10.14).

Further, assume that the operators L_1 and L_2 are both positive definite. Then the sets of eigenvalues $\{\lambda_n^{(1)}\}$ and $\{\lambda_n^{(2)}\}$ are all positive real numbers. Hence, the solutions of Equations (11.51), for arbitrary initial conditions, take one of three forms depending on the sign of the discriminant

$$d_n = \left[\lambda_n^{(1)}\right]^2 - 4\lambda_n^{(2)} \tag{11.52}$$

of Equation (11.51). These forms are described next.

If, for each value of the index n, $d_n > 0$, the solution of Equation (10.14) does not oscillate. In this situation, the structure is said to be *overdamped* and the temporal solutions $a_n(t)$ are all of the form

$$a_n(t) = C_n e^{r_{1n}t} + D_n e^{r_{2n}t} \tag{11.53}$$

where C_n and D_n are constants of integration determined by the initial conditions and r_{1n} and r_{2n} are the positive real roots of the characteristic equation associated with Equation (11.51). In this case, the solution $w(x, t)$ of Equation (10.14) does not oscillate in time.

Also, if the eigenvalues of the operators L_1 and L_2 are such that $d_n = 0$ for each index n, the solution of Equation (11.51) does not oscillate. In this case, the structure is said to be *critically damped*, and the solutions, $a_n(t)$, are all of the form

$$a(t) = (C_n t + D_n)\, e^{-(r_n/2)t} \tag{11.54}$$

where, as before, C_n and D_n are constants of integration determined by the initial conditions and r_n is the positive real repeated root of the characteristic equation associated with Equation (11.51). As in the overdamped case, the solution $w(x,t)$ of Equation (10.14) does not oscillate in time.

If, for each value of the index n, the eigenvalues of L_1 and L_2 are such that $d_n < 0$, then the solution of Equation (11.15) oscillates with decaying amplitude. In this case, the structure is said to be *underdamped*, and the solutions $a_n(t)$ are all of the form

$$a_n(t) = e^{-(\lambda_n/2)t}(C_n \sin \omega_n t + D_n \cos \omega_n t) \tag{11.55}$$

where C_n and D_n are constants of integration determined by the initial conditions. Here, λ_n and ω_n are positive real numbers determined from the roots of the characteristic equation, which appear in complex conjugate pairs. In this case, the solution $w(x,t)$ of Equation (10.14) oscillates in time with decaying amplitude.

An additional possibility is that, for a given structure, d_n takes on different signs. That is, there exists an n for which $d_n < 0$ and at least one value of the index n such that $d_n > 0$. In this situation, the structure is said to be *mixed damped*. The solution $w(x,t)$ will contain one oscillatory term and at least one nonoscillatory term.

The preceding four conditions can be checked and the oscillatory nature of the solution determined without calculating the eigenvalues of the operators L_1 and L_2. The definiteness of the operator $(L_1^2 - 4L_2)$ can be determined by simple integration and, as illustrated by Inman and Andry (1982), the definiteness determines the oscillatory nature of the solution $w(x,t)$. In particular, if L_1 and L_2 commute on $D(L_1) = D(L_2)$, then:

1. The operator $L_1^2 - 4L_2$ is positive definite on $D(L_2)$ if and only if the structure is overdamped.
2. The operator $L_1^2 = 4L_2$ on $D(L_2)$ if and only if the structure is critically damped.
3. The operator $4L_2 - L_1^2$ is positive definite on $D(L_2)$ if and only if the structure is underdamped.
4. The operator $L_1^2 - 4L_2$ is indefinite on $D(L_2)$ if and only if the structure is mixed damped.

These conditions specify the oscillatory nature of the solution $w(x,t)$ of Equation (10.14). In addition to providing a criterion for oscillation, they also lead to simple inequalities in the parameters of the structure, which can be used in design and control applications. The following example illustrates this point.

Example 11.9.1

Consider the longitudinal vibration of a clamped bar with both internal and external damping. Under the assumption of linearity, the equation of motion is written as

$$u_{tt}(x,t) + 2\left[\gamma - b\frac{\partial^2}{\partial x^2}\right]u_t(x,t) - a\frac{\partial^2}{\partial x^2}u(x,t) = 0, \qquad x \in \Omega = (0,1)$$

where $u(x,t)$ is the displacement of the bar, and the positive constants γ, b, and a reflect the relevant physical parameters. Thus, the operator forms are

$$L_1 = 2\left[\gamma - b\frac{\partial^2}{\partial x^2}\right] \qquad \text{and} \qquad L_2 = -a\frac{\partial^2}{\partial x^2}$$

The boundary conditions are taken to be

$$u(0, t) = u(1, t) = 0$$

with $D(L_2) = \{u(\cdot, t) \in \mathcal{L}_2^{\mathrm{R}}(0, 1)$ such that all partial derivatives up to order 4 are in $\mathcal{L}_2^{\mathrm{R}}(0, 1)$ and $u(0, t) = u(1, t) = 0\}$. Note that L_1 and L_2 are positive definite and commute on $D(L_2)$. Also, calculation of L_1^2 and $L_1^2 - 4L_2$ yields

$$L_1^2 = 4\left(\gamma^2 - 2\gamma b\frac{\partial^2}{\partial x^2} + b^2\frac{\partial^4}{\partial x_4}\right)$$

and

$$L_1^2 = 4L_2 = 4\left(\gamma^2 - 2\gamma b\frac{\partial^2}{\partial x^2} + b^2\frac{\partial^4}{\partial x_4} + a\frac{\partial^2}{\partial x^2}\right)$$

The eigenvalues of the operator $L_1^2 - 4L_2$ are then

$$4\left(\gamma^2 - 2\gamma\frac{b}{a}\lambda_n + \frac{b^2}{a^2}\lambda_n^2 - \lambda_n\right) = 4\left[\left(\gamma - \frac{b}{a}\lambda_n\right)^2 - \lambda_n\right]$$

where $\lambda_n = an^2\pi^2$. Demanding that the sign of this expression be positive, negative, or zero is equivalent to characterizing the definiteness of the operator $4L_2 - L_1^2$. Note that the only possibilities for this problem are either overdamping for each mode or mixed damping with all higher modes overdamped.

CHAPTER NOTES

The material of this chapter is a much condensed, and somewhat oversimplified, version of the contents of a course in applied functional analysis. Several texts are recommended and were used to develop the material here. The books by Stakgold (1967, 1968) present most of the material here in two volumes. This text, republished in 2000 (Stakgold, 2000a, 2000b), is recommended because it makes a very useful comparison between finite-dimensional systems (matrices) and infinite-dimensional systems. The book by Hocstadt (1973) presents an introduction to Hilbert spaces and compact operators in fairly short order. A good fundamental text on functional analysis, such as Bachman and Narici (1966), or on operator theory, such as Naylor and Sell (1982), presents the information of Sections 11.2 through 11.5 in rigorous detail. The material of Section 11.5 is usually not presented, except formally in vibration texts. MacCluer (1994) presents conditions for the existence of convergent eigenfuction expansions and connects this to operator properties that result in the successful application of the method of separation of variables.

The eigenvalue estimate methods given in Section 11.7 are the most common eigenvalue estimates and can be found in most texts, including Stakgold (1967, 2000a). The literature is full of improvements and variations of these estimates. The short and excellent text by Weinberger (1974) is summarized in Section 11.8 on enclosure theorems. These theorems are quite useful, but apparently have not been taken advantage of by the engineering community. The oscillation results of Section 11.9 paraphrase the paper by Inman and Andry (1982) and present a direct extension of Section 3.6.

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PROBLEMS

11.1 Show that $\mathcal{L}_2^{\mathrm{R}}(\Omega)$ is a linear space.

11.2 Show that the sequence of example 11.2.1 is in fact a Cauchy sequence.

11.3 Show that the sequence of example 11.2.1 converges to $u(x)$ as claimed.

11.4 Show that the operator of example 11.4.1 is linear.

11.5 For the operator of example 11.4.1, show that $N(L), R(L)$, and $D(L)$ are linear spaces and hence subspaces of $\mathcal{H}$.

11.6 Consider the operator $-\partial^2/\partial x^2$, defined on $\mathcal{L}_2^{\mathrm{R}}(0, \ell)$ such that $u_x(0) = u_x(\ell) = 0$. Calculate the null space of the operator and show that it is a subspace of $\mathcal{H}$. Is this operator equal to the operator of example 11.4.1?

11.7 Show that the linear functional defined by Equation (11.14) is in fact bounded.

11.8 Show that the operator defined by the Green's function of example 10.6.1 is a bounded operator.

11.9 Calculate the adjoint of the operator in problem 11.6. Is the operator formally self-adjoint? Positive definite?

11.10 Show that the operator $-\partial^4/\partial x^4$ defined on $\mathcal{L}_2^{\mathrm{R}}(0, \ell)$ with boundary conditions $u(0) = u(\ell) = u'(0) = u'(\ell) = 0$ is formally self-adjoint.

11.11 Consider the transverse vibrations of a beam with variable stiffness ($EI(x)$) and of dimensions compatible with the Euler–Bernoulli assumptions and with cantilevered boundary conditions. Show that the corresponding operator is symmetric, positive definite, and self-adjoint.

11.12 Calculate the adjoint of the operator $L = \partial/\partial x$ with boundary conditions $u(0) = \alpha u(1)$, α constant. [Do not forget to calculate $D(L^*)$.]

11.13 Suppose that A is a compact linear operator and B is a bounded linear operator such that AB is defined. Show that AB is compact.

11.14 Show that, if the linear self-adjoint operators L_1 and L_2 commute and if L_2 has a compact inverse and L_1 is nonsingular, then L_1 and L_2 have a common set of eigenfunctions.

11.15 Show that the identity operator is not compact.

11.16 Prove that, if L has a compact inverse, it is positive definite if and only if each of its eigenvalues is positive.

11.17 Calculate some estimates of the eigenvalues of the operator for the transverse vibration of a simply supported, nonuniform beam with $EI(x) = (1.1 - x)$. Compare the results of your estimates to the exact values for $EI = 1$.

11.18 Calculate the eigenvalues of a square membrane clamped along its boundary on each of the following:

(a) the square defined by the axis and the line $x = 1$, $y = 1$;
(b) the square defined by the axis and the line $x = 2$, $y = 2$;
(c) the square defined by the axis and the line $x = 3$, $y = 3$.

Compare the results of these three operators using the enclosure result of example 11.8.2 and illustrate that the monotonicity results hold.

11.19 Consider the transverse vibrations of three beams all of dimensions compatible with the Euler–Bernoulli assumptions and all with cantilevered boundary conditions. Suppose two of the beams have constant stiffness denoted by E_1I_1 and E_2I_2 respectively and that the third beam has a variable stiffness denoted by $EI(x)$. Show that, if $E_1I_1 < EI(x) < E_2I_2$, then the eigenvalues of the variable-stiffness beam fall in between those of the constant-stiffness beams.

11.20 Consider the damped plate described by Equation (9.87) with simply supported boundary conditions. Calculate inequalities in the constants ρ, γ, and D_E such that the free response is (a) overdamped, (b) critically damped, and (c) underdamped. What can you conclude from your calculation?

11.21 Consider the problem of example 11.9.1. Can this system be designed to be underdamped if a mass is attached to one end and fixed at the other?

11.22 Show that a differential operator must be of even order for it to be symmetric. (*Hint*: Use integration of parts to show that an odd-order differential operator is not symmetric.)