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12
Forced Response and Control

12.1 INTRODUCTION

This chapter considers the response of distributed-parameter structures that are under some external influence. This includes consideration of the response of distributed-mass structures to applied external forces, the response of distributed-mass structures connected to lumped-mass elements, and the response of distributed-mass structures under the influence of both passive and active control devices.

If the equations of motion can be decoupled, then many of the results used for lumped-mass systems described in Chapter 5 can be repeated for the distributed-mass case. However, because of the infinite-dimensional nature of distributed-mass systems, convergence of solutions occasionally preempts the use of these methods. Convergence issues are especially complicated if the structure is subjected to control forces or unknown disturbances.

12.2 RESPONSE BY MODAL ANALYSIS

This section considers the forced response of damped distributed-parameter systems of Equation (10.14) of the form

$$w_{tt}(x,t) + L_1 w_t(x,t) + L_2 w(x,t) = f(x,t), \qquad x \in \Omega \tag{12.1}$$

with appropriate boundary and initial conditions. Here, the operators L_1 and L_2 are self-adjoint, positive definite operators; L_2 has a compact inverse, and L_1 shares the set of eigenfunctions $\{\Theta_n(x)\}$ with L_2 (i.e., L_1 and L_2 commute). For the moment, the only assumption made of $f(x,t)$ is that it lies in $\mathcal{L}_2^{\mathrm{R}}(\Omega)$.

Since $f(x,t) \in \mathcal{L}_2^{\mathrm{R}}(\Omega)$, Equation (12.1) can be multiplied by the function $\Theta_n(x)$ and then integrated over Ω. This integration yields

$$(w_{tt}, \Theta_n) + (L_1 w_t, \Theta_n) + (L_2 w, \Theta_n) = (f, \Theta_n) \tag{12.2}$$

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The left-hand side of this equation is identical to Equation (10.16). Applying the analysis of Section 11.6, Equation (12.2) becomes

$$\ddot{a}_n(t) + \lambda_n^{(1)}\dot{a}_n(t) + \lambda_n^{(2)}a_n(t) = f_n(t), \qquad n = 1, 2, 3 \tag{12.3}$$

where $f_n(t)$ has the form

$$f_n(t) = \int_\Omega f(x, t)\Theta_n(x)\, d\Omega, \qquad n = 1, 2, 3 \tag{12.4}$$

This scalar equation in the function $a_n(t)$ can be solved and analyzed using the single-degree-of-freedom model of Section 1.4.

Equation (12.3) is essentially the same as Equation (5.38), and the solution is thus given by Equation (5.39). That is, if the system is underdamped ($4L_2 - L_1^2 > 0$), then for zero initial conditions

$$a_n(t) = \frac{1}{\omega_{dn}}\int_0^t e^{-\zeta_n\omega_n\tau} f_n(t-\tau)\sin(\omega_{dn}\tau)\, d\tau \tag{12.5}$$

where for each value of the index n

$$\omega_n = \sqrt{\lambda_n^{(2)}}, \quad \text{the } n\text{th natural frequency} \tag{12.6}$$

$$\zeta_n = \frac{\lambda_n^{(1)}}{2\sqrt{\lambda_n^{(2)}}}, \quad \text{the } n\text{th modal damping ratio} \tag{12.7}$$

$$\omega_{dn} = \omega_n\sqrt{1-\zeta_n^2}, \quad \text{the } n\text{th damped natural frequency} \tag{12.8}$$

Thus, in the solution where the operators L_1 and L_2 commute, the temporal coefficients in the series solution are determined by using results from single-degree-of-freedom theory discussed in Chapter 1. The solution to Equation (12.1) is the sum

$$w(x, t) = \sum_{n=1}^{\infty} a_n(t)\Theta_n(x) \tag{12.9}$$

where the $a_n(t)$ are determined by Equation (12.5) for the case where the initial conditions are set to zero and the set $\{\Theta_n(x)\}$ consists of the eigenfunctions of the operator L_2. Since the set of functions $\{\Theta_n(x)\}$ consists of the modes of free vibration, the procedure just described is referred to as a modal analysis solution of the forced response problem.

Example 12.2.1

Consider the hinged–hinged beam of example 10.3.1. Assuming the beam is initially at rest ($t = 0$), calculate the response of the system to a harmonic force of $\sin(t)$ applied at $x = \ell/2$, where ℓ is the length of the beam. Assume the damping in the beam is of the form $2\alpha w_t(x, t)$, where α is a constant. First, note that the operator

$$L_2 = \frac{EI}{m}\frac{\partial^4}{\partial x^4}$$

has a compact inverse and is self-adjoint and positive definite with respect to the given boundary conditions. Furthermore, the eigenfunctions of the operator L_2 serve as eigenfunctions for the operator $L_1 = 2\alpha I$. Thus, the eigenvalues of the operator $4L_2 - L_1^2$ are (for the given boundary conditions)

$$4\left[n^4\pi^4\frac{EI}{m\ell^4} - \alpha^2\right]$$

which are greater than zero for every value of the index n if

$$\frac{\pi^2}{\ell^2}\sqrt{\frac{EI}{m}} > \alpha$$

Hence, each coefficient, $a_n(t)$, is underdamped in this case. The solution given by Equation (12.5) then applies.

The forcing function for the system is described by

$$f(x, t) = \delta\left(x - \frac{\ell}{2}\right)\sin t$$

where $\delta(x - \ell/2)$ is the Dirac delta function. Substitution of this last expression into Equation (12.4) along with the normalized eigenfunctions of example 10.3.1 yields

$$\begin{aligned} f_n(t) &= \sqrt{\tfrac{2}{\ell}}\sin t\int_0^\ell \sin\left(\tfrac{n\pi x}{\ell}\right)\delta\left(x - \tfrac{\ell}{2}\right)dx \\ &= \sqrt{\tfrac{2}{\ell}}\sin t\sin\tfrac{n\pi}{2} \end{aligned}$$

In addition, the natural frequency, damping ratio, and damped natural frequency become

$$\begin{aligned} \omega_n &= \left(\frac{n\pi}{\ell}\right)^2\sqrt{\frac{EI}{m}} \\ \zeta_n &= \frac{\alpha}{\omega_n} \\ \omega_{dn} &= \omega_n\sqrt{1 - \frac{\alpha^2}{\omega_n^2}} \end{aligned}$$

With these modal damping properties determined, specific computation of Equation (12.5) can be performed. Note that the even modes are not excited in this case, since $f_{2n} = 0$ for each n. Physically, these are zero because the even modes all have nodes at the point of excitation, $x = \ell/2$.

If the damping in the system is such that the system is overdamped or mixed damped, the solution procedure is the same. The only difference is that the form of $a_n(t)$ given by Equation (12.5) changes. For instance, if there is zero damping in the system, then $\zeta_n \to 0$, $\omega \to \omega_n$, and the solution of Equation (12.5) becomes

$$a_n(t) = \frac{1}{\omega_n}\int_0^t f_n(t - \tau)\sin(\omega_n\tau)\,d\tau \tag{12.10}$$

12.3 MODAL DESIGN CRITERIA

The previous section indicates how to calculate the forced response of a given structure to an external disturbance by modal analysis. This modal approach is essentially equivalent to decoupling a partial differential equation into an infinite set of ordinary differential equations. This section examines some of the traditional design formulae for single-degree-of-freedom oscillators applied to the modal coordinates of a distributed-parameter structure of the form given in Equation (12.9). This modal design approach assumes that the summation of Equation (12.9) is uniformly convergent and the set of eigenfunctions $\{\Theta_n(x)\}$ is complete. Hence, there is a value of the index n, say $n = N$, for which the difference between $w(x, t)$ and the partial sum

$$\sum_{n=1}^{N} a_n(t)\Theta_n(x)$$

is arbitrarily small. Physically, observation of certain distributed-mass systems indicates that some key modes seem to dominate the response, $w(x, t)$, of the system. Both the mathematics and the physics in this case encourage the use of these dominant modes in the design criteria.

As an illustration of modal dominance, consider again the problem of example 12.2.1. With zero initial conditions, the response $a_n(t)$ is of the same form as Equation (1.18) multiplied by 0, 1, or -1, depending on the value of n (i.e., $\sin n\pi/2$). In fact, integration of Equation (12.5) for the case $f_n(t) = f_{n0} \sin \omega t$ yields

$$a_n(t) = X_n \sin(\omega t + \beta_n) \tag{12.11}$$

The coefficient X_n is determined (see Section 1.4) to be

$$X_n = \frac{f_{n0}}{\sqrt{(\lambda_n^{(2)} - \omega^2)^2 + (\lambda_n^{(1)}\omega)^2}} \tag{12.12}$$

and the phase shift β_n becomes

$$\beta_n = \tan^{-1} \frac{\lambda_n^{(1)}\omega}{\lambda_n^{(2)} - \omega^2} \tag{12.13}$$

The quantity X_n can be thought of as a *modal participation factor* in that it is an indication of how dominant the nth mode is. For a fixed value of the driving frequency ω, the values of X_n steadily decrease as the index n increases. The modal participation factor decreases unless the driving frequency is close to the square root of one of the eigenvalues of the operator L_2. In this case the modal participation factor for that index may be a maximum. By examining the modal participation factors or modal amplitudes, the designer can determine which modes are of interest or which modes are most important. The following example illustrates this point.

Example 12.3.1

Calculate the modal amplitudes for the clamped beam of example 12.2.1. Note that in this case the driving frequency is 1, i.e., $\omega = 1$. For the sake of simplicity, let $EI = m = \alpha = 1$and $\ell = 2$, so that the system is underdamped. From example 12.2.1, $f_{n0} = 1$ for each n. Also, $\lambda_n^{(1)} = 2$ for each n and

$$\lambda_n^{(2)} = 6.088n^4$$

for each value of the index n. In this case, Equation (12.12) yields

$$X_1 = 0.183, \qquad X_2 = 0.010, \qquad X_3 = 0.002$$

Note that the modal participation factor, X_n, decreases rapidly with increasing n.

Next, consider the same problem with the same physical parameters, except with a new driving frequency of $\omega = 22$. In this case the modal participation factors are

$$\begin{aligned} X_1 &= 0.002, & X_2 &= 0.003 \\ X_3 &= 0.022, & X_4 &= 0.0009 \\ X_5 &= 0.0003, & X_6 &= 0.0001 \end{aligned}$$

This example illustrates that, if the driving frequency is close to a given mode frequency (X_3 in this case), the corresponding modal amplitude will increase in absolute value.

By examining the solution $w(x, t)$ mode by mode, certain design criteria can be formulated and applied. For example, the magnification curve of Figure 1.9 follows directly from Equation (12.12) on a per mode basis. Indeed, all the design and response characterizations of Section 1.4, such as bandwidth and overshoot, can be applied per mode. However, all the design procedures become more complicated because of coefficient coupling between each of the mode equations given by Equation (12.11). While the equations for $a_n(t)$ are decoupled in the sense that each $a_n(t)$ can be solved for independently of each other, the coefficients in these equations will depend on the same physical parameters (i.e., E, I, ρ, m, and so on). This is illustrated in the following example.

Example 12.3.2

Consider the step response of the clamped beam of example 12.2.1. A modal time to peak can be defined for such a system by using Equation (1.27) applied to Equation (12.3). With a proper interpretation of ζ_n and ω_n, the *modal time to peak*, denoted by t_{pn}, is

$$t_{pn} = \frac{\pi}{\omega_n \sqrt{1 - \zeta_n^2}}$$

where $\omega_n = (n^2\pi^2/\ell^2)\sqrt{EI/m}$ and $\zeta_n = \alpha/\omega_n$. Examination of this formula shows that, if E, I, m, and α are chosen so that t_{p2} has a desired value, then $t_{p3}, t_{p4}, \ldots$ are fixed. Thus, the peak time, overshoot, and so on, of a distributed-mass system cannot be independently chosen on a per mode basis even though the governing equations decouple.

12.4 COMBINED DYNAMICAL SYSTEMS

Many systems are best modeled by combinations of distributed-mass components and lumped-mass components. Such systems are called *hybrid systems*, distributed systems with lumped appendages, or *combined dynamical systems*. This section discusses the natural frequencies and mode shapes of such structures and the use of the eigensolution to solve for the forced response of such structures.

As an example of such a system, consider the free vibration of a beam of length ℓ connected to a lumped mass and spring as illustrated in Figure 12.1. The equation of motion of the beam with the effect of the oscillator modeled as an external force, $f(t)\delta(x-x_1)$, is

$$EIw_{xxxx} + \rho A w_{tt} = f(t)\delta(x - x_1), \qquad x \in (0, \ell) \tag{12.14}$$

The equation of motion of the appended system is given by

$$m\ddot{z}(t) + kz(t) = -f(t) \tag{12.15}$$

where m is the appended mass and k is the associated stiffness. Here, the coordinate, $z(t)$, of the appended mass is actually the displacement of the beam at the point of attachment, i.e.,

$$z(t) = w(x_1, t) \tag{12.16}$$

Combining Equations (12.14) and (12.15) yields

$$\left[EI\frac{\partial^4}{\partial x^4} + k\delta(x - x_1)\right] w(x, t) + [\rho A + m\delta(x - x_1)]\, w_{tt}(x, t) = 0 \tag{12.17}$$

The solution $w(x, t)$ is now assumed to separate, i.e., $w(x, t) = u(x)a(t)$. Following the method of separation of variables, substitution of the separated form into Equation (12.17) and rearrangement of terms yields

$$\frac{EIu''''(x) + k\delta(x - x_1)u(x)}{[\rho A + m\delta(x - x_1)]u(x)} = -\frac{\ddot{a}(t)}{a(t)} \tag{12.18}$$

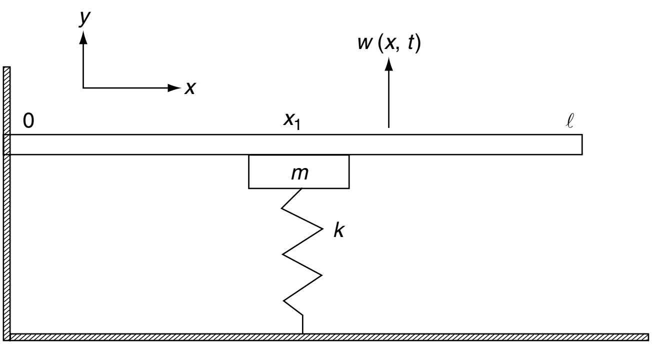

Figure 12.1 A beam with an attached lumped mass–spring system.

As before (see Section 9.2), each side of the equality must be constant. Taking the separation constant to be ω^2, the temporal function $a(t)$ has the harmonic form

$$a(t) = A\sin(\omega t) + B\cos(\omega t) \tag{12.19}$$

where A and B are constants of integration determined by the initial conditions. The spatial equation becomes

$$ELu''''(x) + \left[(k - m\omega^2)\delta(x - x_1) - \rho A\omega^2\right]u(x) = 0 \tag{12.20}$$

subject to the appropriate boundary conditions.

Solution of Equation (12.20) yields the generalized eigenfunctions, $\Theta_n(x)$, and eigenvalues, ω_n^2, for the structure. These are called generalized eigenfunctions because Equation (12.20) does not formally define an operator eigenvalue problem, as specified in Section 10.2. Hence, the procedure and modal analysis are performed formally. Note, however, that, if $k/m = \omega_n^2$, i.e., if the appended spring–mass system is tuned to a natural frequency of the beam, the related eigenfunction becomes that of the beam without the appendage. The solution of Equation (12.20) can be constructed by use of a Green's function for the vibrating beam.

The Green's function $g(x, x_1)$ for a beam satisfies

$$g'''' - \beta^4 g = \delta(x - x_1) \tag{12.21}$$

where $\beta^4 = \rho A\omega^2/(EI)$ and g satisfies the appropriate boundary conditions. Following the development of Section 10.6, Equation (12.21) has the solution

$$g(x, x_1) = -\frac{1}{2\beta^3 \sin\beta\ell \sinh\beta\ell}\begin{cases} y(x, x_1), & 0 < x < x_1 \\ y(x_1, x), & x_1 < x < \ell \end{cases} \tag{12.22}$$

where the function $y(x, x_1)$ is symmetric in x_1 and x and has the form

$$\begin{aligned} y(x, x_1) = {} & \sin(\beta\ell - \beta x_1)\sin(\beta x)\sinh(\beta\ell) \\ & - \sinh(\beta\ell - \beta x_1)\sinh(\beta x)\sin(\beta\ell) \end{aligned} \tag{12.23}$$

In terms of the Green's function just defined, the solution to Equation (12.20) for the simply supported case can be written as (see Nicholson and Bergman, 1986)

$$u(x) = \frac{1}{EI}(m\omega^2 - k)g(x, x_1)u(x_1) \tag{12.24}$$

If $u(x_1)$ were known, then Equation (12.24) would specify the eigenfunctions of the system. Fortunately, the function $u(x_1)$ is determined by writing Equation (12.24) for the case $x = x_1$, resulting in

$$\left[EI - \left\{m\left(\omega^2 - \frac{k}{m}\right)\right\} g\,(x_1, x_1)\right] u(x_1) = 0 \tag{12.25}$$

which yields the characteristic equation for the system. In order to allow $u(x_1)$ to be nonzero, the coefficient in Equation (12.25) must vanish, yielding an expression for computing the natural frequencies, ω. This transcendental equation in ω contains terms of the form $\sin \beta\ell$ and hence has an infinite number of roots, denoted by ω_n. Thus, Equation (12.24) yields an infinite number of eigenfunctions, denoted by $u_n(x)$.

Both the free and forced response of a combined dynamical system, such as the one described in Figure 12.1, can be calculated using a modal expansion for a cantilevered beam. Following Section 9.4, the eigenfunctions are, from Equation (12.20), those of a nonappended cantilevered beam, i.e.,

$$\Theta_i(x) = \cosh \beta_i x - \cos \beta_i x - \alpha_i(\sinh \beta_i x - \sin \beta_i x) \tag{12.26}$$

Here, the constants α_i are given by

$$\alpha_i = \frac{\cosh \beta_i \ell + \cos \beta_i \ell}{\sinh \beta_i \ell + \sin \beta_i \ell} \tag{12.27}$$

and the eigenvalues β_i are determined from the transcendental equation

$$1 + \cosh \beta_i \ell \cos \beta_i \ell = 0 \tag{12.28}$$

Note that in this case the arguments of Section 11.6 hold and the functions $\Theta_i(x)$ form a complete orthogonal set of functions. Hence, the spatial solution $u(x)$ can be written as

$$u(x) = \sum_{i=1}^{\infty} b_i \Theta_i(x) \tag{12.29}$$

with the set $\{\Theta_i\}$ normalized so that

$$(\Theta_i, \Theta_j) = \ell \delta_{ij} \tag{12.30}$$

Substitution of Equation (12.29) for $u(x)$ in Equation (12.20), multiplying by $\Theta_j(x)$, using the property

$$\Theta_j''''(x) = \beta_j^4 \Theta_j(x)$$

and integrating over the interval $(0,\ell)$ yields

$$\begin{aligned} EI\ell\beta_i^4 b_i + (k - m\omega^2)\Theta_i(x_1) b_i \Theta_i(x_1) - \rho A \ell \omega^2 b_i = 0 \qquad & \text{for } i = j \\ (k - m\omega^2)\Theta_j(x_1) b_i \Theta_i(x_1) = 0 \qquad & \text{for } i \neq j \end{aligned} \tag{12.31}$$

Dividing this last expression by $\rho A \ell$ and defining two new scalars, A_{ij} and B_{ij}, by

$$A_{ij} = \frac{k\Theta_i(x_1)\Theta_j(x_1)}{\rho A \ell} + \frac{EI\beta_i^4}{\rho A}\delta_{ij} \tag{12.32}$$

$$B_{ij} = \frac{1}{\rho A \ell} m \Theta_i(x_1)\Theta_j(x_1) + \delta_{ij} \tag{12.33}$$

allows Equation (12.31) to be simplified. Equation (12.31) can be rewritten as

$$\sum_{j=1}^{\infty} A_{ij} b_j = \omega^2 \sum_{j=1}^{\infty} B_{ij} b_j \tag{12.34}$$

This last expression is in the form of a generalized infinite matrix eigenvalue problem for ω^2 and the generalized Fourier coefficients b_j. The elements b_j are the modal participation factors for the modal expansion given by Equation (12.29).

The orthogonality relationship for the eigenfunctions $\{\Theta_n(x)\}$ is calculated from Equation (12.20) and rearranged in the form

$$\Theta_n''''(x) - \left(\frac{\rho A \omega^2}{EI}\right)\Theta_n(x) = \frac{1}{EI}(\omega_n^2 m - k)\delta(x - x_1)\Theta_n(x) \tag{12.35}$$

Premultiplying Equation (12.35) by $\Theta_m(x)$ and integrating yields (see problem 12.5)

$$\rho A \int_0^{\ell} \Theta_m(x)\Theta_n(x)\,dx = -\int_0^{\ell} m\delta(x - x_1)\Theta_m(x)\Theta_n(x)\,dx$$

or

$$\int_0^{\ell} \left[1 + \frac{m}{A\rho}\delta(x - x_1)\right]\Theta_m(x)\Theta_n(x)\,dx = \delta_{nm} \tag{12.36}$$

The preceding characteristic equation and orthogonality relationship completes the modal analysis of a cantilevered Euler–Bernoulli beam connected to a spring and lumped mass.

Equipped with the eigenvalues, eigenfunctions, and the appropriate orthogonality condition, a modal solution for the forced response of a damped structure can be carried out for a proportionally damped beam connected to a lumped spring–mass dashpot arrangement following these procedures. Bergman and Nicholson (1985) showed that the modal equations for a damped cantilevered beam attached to a spring–mass dashpot appendage have the form

$$\ddot{a}_n(t) + \sum_{m=1}^{\infty}\left\{\varepsilon_b\delta_{nm} + \mu\left(\frac{\varepsilon\alpha_m^4\alpha_n^4}{\alpha_0^8} - \varepsilon_b\right)A_m A_n \Theta_m(x_1)\Theta_n(x_1)\dot{a}_m(t)\right\} + \alpha_n^4 a_n(t) = f_n(t) \tag{12.37}$$

Here

$$f_n(t) = \gamma\int_0^L \Theta_n(x) f(x, t)\,dx$$

$f(x, t) =$ externally applied force

$\varepsilon_b =$ distributed damping coefficient

$\varepsilon =$ lumped damping rate

$\mu =$ lumped mass

$\alpha_n^2 =$ system natural frequencies

$\alpha_0 =$ lumped stiffness

$$A_n = \frac{\alpha_0^4}{\alpha_0^4 - \alpha_n^4}$$

With the given parameters and orthogonality conditions, the combined system has a modal solution given by

$$w(x,t) = \sum_{n=1}^{\infty} a_n(t)\Theta_n(x) \tag{12.38}$$

where $a_n(t)$ satisfies Equation (12.37) and the appropriate initial conditions. Note from Equation (12.37) that proportional damping results if $\varepsilon\alpha_m^4\alpha_n^4 = \varepsilon_b\alpha_0^8$.

12.5 PASSIVE CONTROL AND DESIGN

The lumped appendage attached to a beam of the previous section can be viewed as a passive control device, much in the same way that the absorber of Section 6.2 can be thought of as a passive control element. In addition, the layered materials of Section 9.6 can be thought of as either a passive control method or a redesign method. In either case, the desired result is to choose the parameters of the system in such a way that the resulting structure has improved vibration response.

First, consider a single absorber added to a cantilevered beam. The equations of motion as discussed in the previous section have a temporal response governed by Equation (12.37). Thus, the rate of decay of the transient response is controlled by the damping terms:

$$\sum_{m=1}^{\infty} \left\{ \varepsilon_b\delta_{nm} + \mu\frac{\varepsilon\alpha_m^4\alpha_n^4}{\alpha_0^8} - \varepsilon_b A_m A_n \Theta_m(x_1)\Theta_n(x_1) \right\} \dot{a}_m(t) \tag{12.39}$$

The design problem becomes that of choosing x_1, ε, μ, and α_0 so that Equation (12.39) has the desired value. With only four parameters to choose and an infinite number of modes to effect, there are not enough design parameters to solve the problem. In addition, the summation in Equation (12.39) effectively couples the design problem so that passive control cannot be performed on a per mode basis. However, for specific cases the summation can be truncated, making the design problem more plausible.

Next, consider the layered material of Section 9.6. Such materials can be designed to produce both a desired elastic modulus and a desired loss factor (Nashif, Jones, and Henderson, 1985). Consider the problem of increasing the damping in a beam so that structural vibrations in the beam decay quickly. Researchers in the materials area often approach the problem of characterizing the damping in a material by using the concept of loss factor, introduced as η in Section 1.4, and the concept of *complex modulus* introduced next.

For a distributed-mass structure, it is common practice to introduce damping in materials by simply replacing the elastic modulus for the material, denoted by E, with a complex modulus of the form

$$E(1 + \mathrm{i}\eta) \tag{12.40}$$

where η is the experimentally determined loss factor for the material and i is the square root of (-1). The rationale for this approach is based on an assumed temporal solution of the form $A\,e^{i\omega t}$. If $A\,e^{i\omega t}$ is substituted into the equation of motion of a damped structure, the velocity term yields a coefficient of the form $\mathrm{i}\omega$, so that the resulting equation may be viewed as having a complex stiffness. This form of damping is also called the *Kimball–Lovell complex stiffness* (see Bert, 1973).

The loss factor for a given structure made of standard metal is usually not large enough to suppress unwanted vibrations in many applications. One approach to designing more highly damped structures is to add a layer of damping material to the structure, as indicated in Figure 12.2. The new structure then has different elastic modulus (frequencies) and loss factor. In this way, the damping material can be thought of as a passive control device used to change the poles of an existing structure to more desirable locations. Such a treatment of structures is called *extensional damping*. Sometimes it is referred to as *unconstrained layer damping*, or *free layer damping*.

Let E and η denote the elastic modulus and loss factor of the combined system of Figure 12.2. Let E_1 and η_1 denote the modulus and loss factor of the original beam, and let E_2 and η_2 denote the modulus and loss factor of the added damping material. In addition, let H_2 denote the thickness of the added damping layer and H_1 denote the thickness of the original beam. Let $e_2 = E_2/E_1$ and $h_2 = H_2/H_1$. The design formulae relating the 'new' modulus and loss factor to those of the original beam and added damping material are given in Nashif, Jones, and Henderson (1985) as

$$\frac{EI}{E_1 I_1} = \frac{1 + 4e_2h_2 + 6e_2h_2^2 + 4e_2h_2^3 + e_2^2h_2^4}{1 + e_2h_2} \tag{12.41}$$

and

$$\frac{\eta}{\eta_1} = \frac{e_2h_2(3 + 6h_2 + 4h_2^2 + 2e_2h_2^3 + e_2^2h_2^4)}{(1 + e_2h_2)(1 + 4e_2h_2 + 6e_2h_2^2 + 4e_2h_2^3 + e_2^2h_2^4)} \tag{12.42}$$

where $(e_2h_2)^2$ is assumed to be much smaller than e_2h_2.

Equations (12.41) and (12.42) can be used to choose an appropriate damping material to achieve a desired response.

The preceding complex modulus approach can also be used to calculate the response of a layered structure. Note that the response of an undamped uniform beam can be written in the form

$$w(x, t) = \sum_{n=1}^{\infty} g(E)a_n(t, E)\Theta_n(x, E) \tag{12.43}$$

where $g(E)$ is some function of the modulus E. This functional dependence is usually not explicitly indicated but rather is contained in the eigenvalues of the eigenfunctions

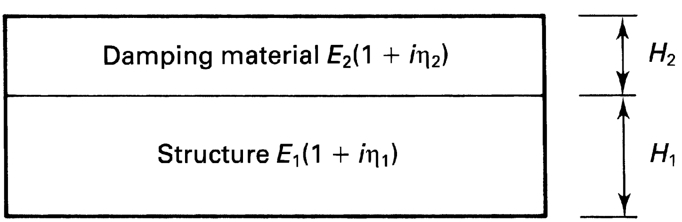

Figure 12.2 Passive vibration control by using a damping layer.

$\Theta_n(x, E)$ and in the temporal coefficients $a_n(t, E)$. One approach used to include the effects of damping in a layered beam is simply to substitute the values of $E + \mathrm{i}\eta$ obtained from Equations (12.41) and (12.42) into g, a_n, and Θ_n in Equation (12.43). Each term of the series (12.43) is complex and of the form $g(E(1+\mathrm{i}\eta)a_n(t, E(1+\mathrm{i}\eta))\Theta_n(x, E(1+\mathrm{i}\eta))$, so some manipulation is required to calculate the real and imaginary parts. This approach should be treated as an approximation, as it is not rigorous.

12.6 DISTRIBUTED MODAL CONTROL

In this section, the control of systems governed by partial differential equations of the form of Equation (12.1) is considered. The control problem is to find some function $f(x, t)$ such that the response $w(x, t)$ has a desired form. If $f(x, t)$ is a function of the response of the system, then the resulting choice of $f(x, t)$ is called active control. If, on the other hand, $f(x, t)$ is thought of as a change in the design of the structure, it is referred to as passive control (or redesign). Modal control methods can be used in either passive or active control. Any control method that uses the eigenfunctions, or modes, of the system in determining the control law $f(x, t)$ is considered to be a modal control method.

Repeating the analysis of Section (12.2) yields the modal control equations. For the control problem, the functions $f_n(t)$ of Equation (12.4) are thought of as modal controls, or inputs, in the jargon of Chapter 7. As indicated in Chapter 7, there are many possible control techniques to apply to Equation (12.3). Perhaps the simplest and most physically understood is state feedback. Viewed by itself, Equation (12.3) is a two-state model, with the states being the generalized velocity, $\dot{a}_n(t)$, and position, $a_n(t)$. If $f_n(t)$ is chosen in this way, Equation (12.3) becomes

$$\ddot{a}_n(t) + \lambda_n^{(1)}\dot{a}_n(t) + \lambda_n^{(2)}a_n(t) = -c_n^p a_n(t) \; - \; c_n^v \dot{a}_n(t) \tag{12.44}$$

where c_n^p and c_n^v are modal position and velocity gains respectively. Obviously, the choice of the position and velocity feedback gains completely determines the nth temporal coefficient in the free response given by Equation (12.10). In theory, c_n^p and c_n^v can be used to determine such performance criteria as the overshoot, decay rate, speed of response, and so on. These coefficients can be chosen as illustrated for the single-degree-of-freedom problem of example 12.6.1.

The question arises, however, about the convergence of $f(x, t)$. Since

$$f_n(t) = -c_n^p a_n(t) - c_n^v \dot{a}_n(t) = \int_\Omega f(x, t)\Theta_n(x)\, d\Omega \tag{12.45}$$

the series

$$f(x, t) = \sum_{n=1}^{\infty} [-c_n^p a_n(t) \; - \; c_n^v \dot{a}_n(t)]\, \Theta_n(x)\, d\Omega \tag{12.46}$$

must converge. Furthermore, it must converge to some function $f(x, t)$ that is physically realizable as a control. Such controls $f(x, t)$ are referred to as *distributed controls* because they are applied along the spatial domain Ω.

Example 12.6.1

Consider the problem of controlling the first mode of a flexible bar. An internally damped bar clamped at both ends has equations of motion given by Equation (12.1) with

$$L_1 = -2b\frac{\partial^2}{\partial x^2}, \qquad L_2 = -\alpha\frac{\partial^2}{\partial x^2}$$

and boundary conditions $w(x, t) = 0$ at $x = 0$ and $x = 1$. Here, b is the constant denoting the rate of internal damping, and α denotes a constant representing the stiffness in the bar (EI/ρ). Solution of the eigenvalue problem for L_1 and L_2 and substitution of the appropriate eigenvalues into Equation (12.3) yields

$$\ddot{a}_n(t) + 2bn^2\pi^2\dot{a}_n(t) + \alpha n^2\pi^2 a_n(t) = f_n(t)$$

For the sake of illustration, assume that $\alpha = 400\pi^2$ and $b = 1$ in the appropriate units.

Suppose it is desired to control only the lowest mode. Furthermore, suppose it is desired to shift the frequency and damping ratio of the first mode. Note that the equation for the temporal coefficient for the first mode is

$$\ddot{a}_1(t) + 2\pi^2\dot{a}_n(t) + 400\pi^4 a_1(t) = f_1(t)$$

so that the first mode has an undamped natural frequency of $\omega_1 = 20\pi^2$ and a damping ratio of $\zeta_1 = 0.05$.

The control problem is taken to be that of calculating a control law, $f(x, t)$, that raises the natural frequency to $25\pi^2$ and the damping ratio to 0.1. This goal will be achieved if the displacement coefficient, after control is applied, has the value

$$(25\pi^2)^2 = 624\pi^4$$

and the velocity coefficient of the closed-loop system has the value

$$2\zeta_1\omega_1 = 2(0.1)(25\omega^2) = 5\omega^2$$

Using Equation (12.44) with $n = 1$ yields

$$\ddot{a}_1(t) + 2\pi^2\dot{a}_n(t) + 400\pi^4 a_1(t) = -c_1^p a_1(t) - c_1^v\dot{a}_1(t)$$

Combining position coefficients and then velocity coefficients yields the following two simple equations for the control gains:

$$\begin{aligned} c_1^p + 400\pi^4 &= 625\pi^4 \\ c_1^v + 2\pi^2 &= 5\pi^2 \end{aligned}$$

Thus, $c_1^p = 225\pi^4$ and $c_1^v = 3\pi^2$ will yield the desired first mode values. The modal control force is thus

$$f_1(t) = -225\pi^4 a_1(t) - 3\pi^2\dot{a}_1(t)$$

In order to apply this control law only to the first mode, the control force must be of the form

$$f(x, t) = f_1(t)\Theta_1(x)$$

For this choice of $f(x, t)$ the other modal controls, $f_n(t),\ n > 1$, are all zero, which is very difficult to achieve experimentally because the result requires $f(x, t)$ to be distributed along a single mode.

Unfortunately, designing distributed actuators is difficult in practice. The design of actuators that produce a spatial distribution along a given mode, as required by the example, is even more difficult. Based on the availability of actuators, the more practical approach is to consider actuators that act at a point, or points, in the domain of the structure. The majority of control methods used for distributed-parameter structures involve using finite-dimensional models of the structure. Such models are often obtained by truncating the series expansion of the solution of Equation (12.9). The methods of Chapter 7 are then used to design a vibration control system for the structure. The success of such methods is tied to the process of truncation (Gibson, 1981). Truncation is discussed in more detail in Chapter 13.

12.7 NONMODAL DISTRIBUTED CONTROL

An example of a distributed actuator that provides a nonmodal approach to control is the use of a piezoelectric polymer. Piezoelectric devices offer a convenient source of distributed actuators. One such actuator has been constructed and used for vibration control of a beam (Bailey and Hubbard, 1985) and is presented here.

Consider the transverse vibrations of a cantilevered beam of length ℓ with a piezoelectric polymer bonded to one side of the beam. The result is a two-layer material similar to the beam illustrated in Figure 12.2. Bailey and Hubbard (1985) have shown that the equation governing the two-layer system is

$$\frac{\partial^2}{\partial x^2}\left[EI\frac{\partial^2 w}{\partial x^2}\right]+\rho A\frac{\partial^2}{\partial t^2}=0, \qquad x\in\Omega \tag{12.47}$$

with boundary conditions

$$\begin{gathered} w(0,t)=w_x(0,t)=0 \\ EIw_{xx}(\ell,t)=-cf(t) \qquad \text{and} \qquad w_{xxx}(\ell,t)=0 \end{gathered} \tag{12.48}$$

where it is assumed that the voltage applied by the polymer is distributed evenly along x, i.e., that its spatial dependence is constant. Here, EI reflects the modulus and inertia of both the beam and the polymer, ρ is the density, and A is the cross-sectional area. The constant c is the bending moment per volt of the material and $f(t)$ is the voltage applied to the polymer. This distributed actuator behaves mathematically as a *boundary control*.

One approach to solving this control problem is to use a Lyapunov function, $V(t)$, for the system, and choose a control, $f(t)$, to minimize the time rate of change of the Lyapunov function.

The chosen Lyapunov function is

$$V(t)=\frac{1}{2}\int_0^\ell\left[\left(\frac{\partial^2 w}{\partial x^2}\right)^2+\left(\frac{\partial w}{\partial t}\right)^2\right]dt \tag{12.49}$$

which is a measure of how far the beam is from its equilibrium position. Minimizing the time derivative of this functional is then equivalent to trying to bring the system to rest (equilibrium) as fast as possible. Differentiating Equation (12.49) and substitution of Equation (12.47) yields

$$\frac{\partial V}{\partial t} = \int_0^\ell \left[1 - \frac{EI}{\rho A}\right] w_{xxt} w_{xx}\, dx + \frac{c}{\rho A} f(t) w_{xx}(\ell, t) \tag{12.51}$$

The voltage $f(t)$ is thus chosen to minimize this last quantity. Bailey and Hubbard (1985) showed that $f(t)$, given by

$$f(t) = -\mathrm{sgn}(c w_{xt}(\ell, t)) f_{\max} \tag{12.52}$$

is used as a minimizing control law. Here, sgn denotes the signum function.

Not only is the control law of Equation (12.52) distributed and independent of the modal description of the structure but it also allows the control force to be magnitude limited, i.e., $|f(t)| < f_{\max}$. These are both very important practical features. In addition, the control law depends only on feeding back the velocity of the tip of the beam. Hence, this distributed control law requires that a measurement be taken at a single point at the tip ($x = \ell$).

12.8 STATE-SPACE CONTROL ANALYSIS

This section examines the control problem for distributed-parameter structures cast in the state-space formulation. Considering Equation (12.1), define the two-dimensional vector $\boldsymbol{z}(x, t)$ by

$$\boldsymbol{z}(x, t) = [\boldsymbol{w}(x, t) \quad \boldsymbol{w}_t(x, t)]^T \tag{12.53}$$

The state equation for the system of Equation (12.2) then becomes

$$\boldsymbol{z}_t = A\boldsymbol{z} + bu \tag{12.54}$$

where the matrix of operators A is defined by

$$A = \begin{bmatrix} 0 & I \\ -L_2 & -L_1 \end{bmatrix} \tag{12.55}$$

the vector b is defined by $b = [0 \quad 1]^T$, and $u = u(x, t)$ is now used to denote the applied force, which in this case is a control. As in the lumped-mass case, there needs to be an observation equation, denoted here as

$$y(x, t) = C\boldsymbol{z}(x, t) \tag{12.56}$$

In addition, the state vector $\boldsymbol{z}$ is subject to boundary conditions and initial conditions and must have the appropriate smoothness (i.e., the elements z belong to a specific function space). Equations (12.55) and (12.56) form the state-space equations for the control of distributed-mass systems and are a direct generalization of Equations (7.1) and (7.2) for the control of lumped-mass systems.

As discussed in the preceding sections, the input or control variable $u(x, t)$ can be either distributed or lumped in nature. In either case the general assumption is that the function $u(x, t)$ separates in space and time. The most common form taken by the control $u(x, t)$ describes the situation in which several time-dependent control forces are applied at various points in the domain of the structure. In this case

$$u(x, t) = \sum_{i=1}^{m} \delta(x - x_i) u_i(t) \tag{12.57}$$

where the m control forces of the form $u_i(t)$ are applied to the m locations x_i. Note that this formulation is consistent with the development of combined dynamical systems of Section 12.4.

The output, or measurement, of the system is also subject to the physical constraint that most devices are lumped in nature. Measurements are most often proportional to a state or its derivatives. In this case, $y(x, t)$ takes the form

$$y(x, t) = \sum_{j=1}^{p} c_j \delta(x - x_j) z(x, t) \tag{12.58}$$

where c_j are measurement gains of each of the p sensors located at the p points, x_j, in the domain of the structure.

The concepts of controllability and observability are of course equally as critical for distributed-mass systems as they are for lumped-mass systems. Unfortunately, a precise definition and appropriate theory is more difficult to develop and hence is not covered here. Intuitively, however, the actuators and sensors should not be placed on nodes of the vibrational modes of the structures. If this practice is adhered to, then the system will be controllable and observable. This line of thought leads to the idea of modal controllability and observability (see, for instance, Goodson and Klein, 1970).

An optimal control problem for a distributed-parameter structure can be formulated following the discussion in Section 7.4 by defining various cost functionals. In addition, pole placement and state feedback schemes can be devised for distributed-parameter systems, generalizing the approaches used in Chapter 7. Note, however, that not all finite-dimensional control methods have direct analogs in distributed-parameter systems. This lack of analogy is largely due to the difference between functional analysis and linear algebra.

CHAPTER NOTES

This chapter discusses the analysis of the forced response and control of structures with distributed mass. Section 12.2 presents standard, well-known modal analysis of the forced response of a distributed-mass system. Such an approach essentially reduces the distributed-mass formulation to a system of single-degree-of-freedom models that can be analyzed by the methods of Chapter 1. Section 12.3 examines some design specifications for distributed-mass systems in modal coordinates. One cannot assign design criteria to each mode independently, as is sometimes suggested by using modal coordinates.

Section 12.4 examines a method of calculating the response of hybrid systems, i.e., systems composed of both distributed-mass elements and lumped-mass elements. Several authors have approached this problem over the years. Most recently, Nicholson and Bergman (1986) produced a series of papers (complete with computer code) discussing combined dynamical systems based on beam equations. Their paper is essentially paraphrased in Section 12.4. Banks *et al.* (1998) clarify the existence of normal modes in combined dynamical systems. The papers by the Bergman group also contain an excellent bibliography of this area. The importance of the analysis of combined dynamical systems is indicated in Section 12.5 on passive control. The most common passive vibration suppression technique is to use an absorber or an isolator. The theory by Bergman *et al.* provides an excellent analytical tool for the design of such systems. An alternative and very useful approach to vibration design and passive control is to use layers of material and high damping (loss factor). This approach is discussed extensively in the book by Nashif, Jones, and Henderson (1985), which provides a complete bibliography.

Section 12.6 discusses the concept of modal control for distributed-mass structures with distributed actuators and sensors. This material is expanded in a paper by Inman (1984). The details of using a modal control method are outlined by Meirovitch and Baruh (1982) and were originally introduced by Gould and Murray-Lasso (1966). Gibson (1981) discusses some of the problems associated with using finite-dimensional state models in designing control laws for distributed-mass systems. This result has sparked interest in nonmodal control methods, an example of which is discussed in Section 12.7. The material of Section 12.7 is taken from the paper by Bailey and Hubbard (1985).

Section 12.8 presents a very brief introduction to formulating the control problem in the state space. Several books, notably Komkov (1970) and Lions (1972), discuss this topic in more detail. A more practical approach to the control problem is presented in the next chapter. Tzou and Bergman (1998) present a collection of works on the vibration and control of distributed-mass systems.

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PROBLEMS

12.1 Calculate the response of the first mode of a clamped membrane of Equation (9.86) subject to zero initial conditions and an applied force of

$$f(x, y, t) = 3\sin t\delta(x - 0.5)\delta(y - 0.5)$$

12.2 Derive the modal response [i.e., $a_n(t)$] for the general system given by Equation (12.1) and associated assumptions if, in addition to $f(x, t)$, the system is subject to initial conditions of the form

$$w(x, 0) = w_0, \qquad w_t(x, 0) = w_{t0}$$

Use the notation of Equations (12.3) through (12.8).

12.3 Calculate an expression for the modal participation factor for problem (12.1).

12.4 Define a modal logarithmic decrement for the system of Equation (12.1) and calculate a formula for it.

12.5 Derive Equation (12.36) from Equation (12.35) by performing the suggested integration. Integrate the term containing Θ_n'''' 4 times using the homogeneous boundary conditions and again using Equation (12.35) to evaluate Θ_m''''.

12.6 Discuss the possibility that the sum in Equation (12.37) can be truncated because

$$\varepsilon\alpha_m^4\alpha_n^4 = \alpha_0^8\varepsilon_b$$

for some choices of m and n.

12.7 Show that the complex stiffness is a consistent representation of the equation of motion by substituting the assumed solution $A\,e^{i\omega t}$ into Equation (10.14). What assumption must be made on the operators L_1 and L_2?

12.8 **(a)** Calculate the terms $g(E)$, $a_n(t, E)$, and $\Theta_n(t, E)$ explicitly in terms of the modulus E for a damped free beam of unit length.

(b) Next, substitute $e(1 + i\eta)$ for E in your calculation and compare your result with the same beam having a damping operator of $L_1 = 2\eta I$, where I is the identity operator.

12.9 Formulate an observer equation for a beam equation using the state-space formulation of Section 12.8.

12.10 Consider the transverse vibration of a beam of length ℓ, modulus E, and mass density ρ. Suppose an accelerometer is mounted at the point $x = \ell/2$. Determine the observability of the first three modes.