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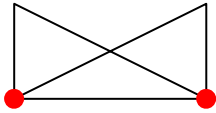
سایت آموزش مهندسی مکانیک ایران

Lecture 4

Announcement: ABAQUS tutorial will be given between 6-8 PM on September 18 (Monday).

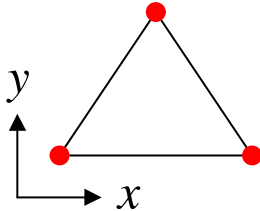
FEM interpolation functions:

1D FEM:



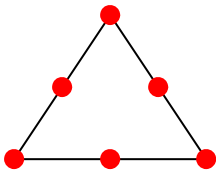
$$w(x) = a_1 + a_2x \quad (\text{linear interpolation})$$

Element types in 2D:



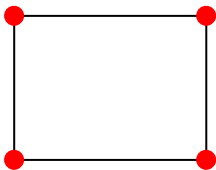
3 noded triangle (linear element)

$$w(x, y) = a_1 + a_2x + a_3y$$



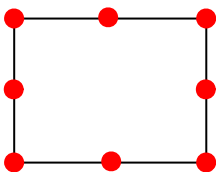
6 noded triangle

$$w(x, y) = a_1 + a_2x + a_3y + a_4xy + a_5x^2 + a_6y^2$$



4 noded quadrilateral

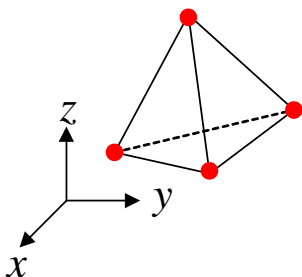
$$w(x, y) = a_1 + a_2x + a_3y + a_4xy$$



8 noded quadrilateral

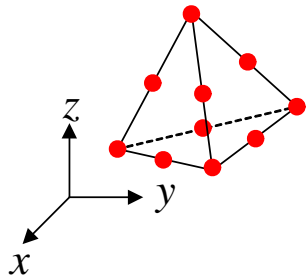
$$w(x, y) = a_1 + a_2x + a_3y + a_4xy + a_5x^2 + a_6y^2 + a_7x^2y + a_8xy^2$$

Element types in 3D:



4 noded tetrahedron (linear element)

$$w(x, y, z) = a_1 + a_2x + a_3y + a_4z$$

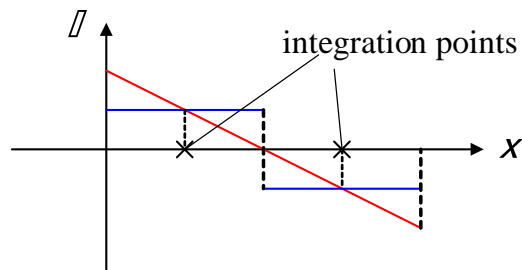


10 noded tetrahedron

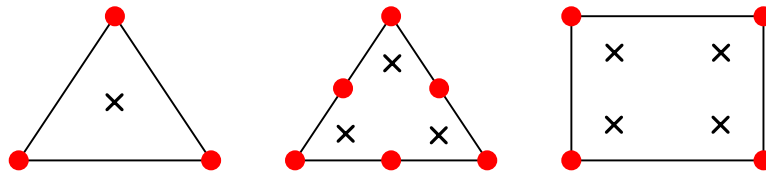
$$w(x, y, z) = a_1 + a_2x + a_3y + a_4z + a_5xy + a_6yz + a_7xz + a_8x^2 + a_9y^2 + a_{10}z^2$$

Integration points for stress:

1D:

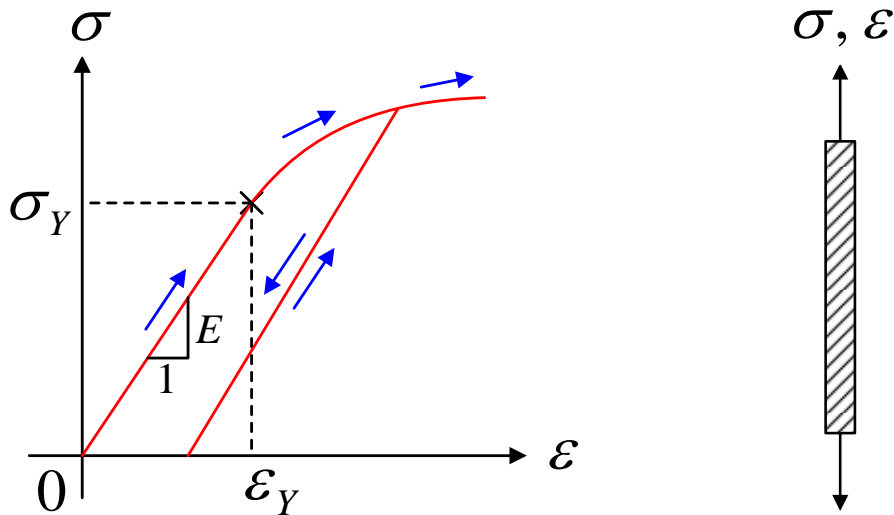


2D:



Material behavior

$\sigma = E\varepsilon$ (Hooke's law is valid before materials yield at a critical strain beyond which irreversible plastic deformation begins to accumulate)



Typically:

Yield strain: $\varepsilon_Y \cong 0.2\%$

Young's modulus: $E \sim 10^{11} \text{ N/m}^2 = 100 \text{ GPa}$ (210 GPa for Steel)

Poisson's ratio: $\nu \sim 1/4 \sim 1/3$ (1/3 for steel)

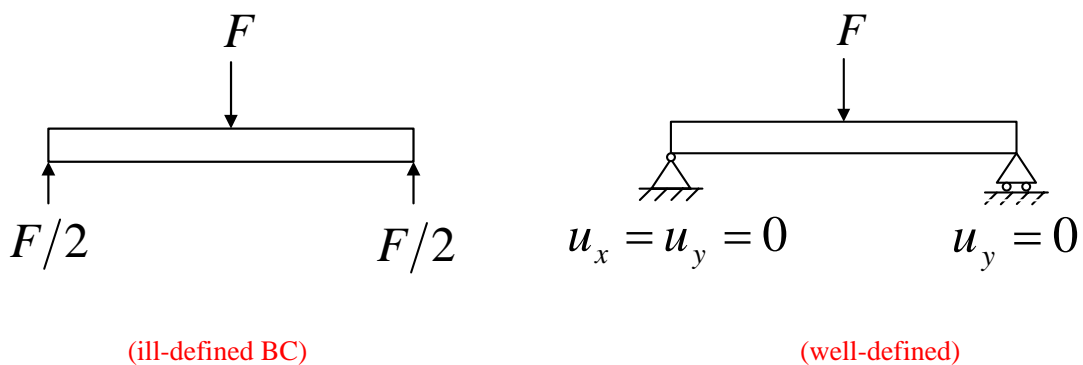
$$\mathbf{M}\ddot{\mathbf{U}} + \mathbf{K}\mathbf{U} = \mathbf{F}$$

\mathbf{M} , \mathbf{K} are positive definite matrices. In numerical calculations such as FEM, it is very important to eliminate rigid body motion. Otherwise, the stiffness matrix will have zero eigenvalues and becomes non-positive definite. This is especially problematic for static problems since we are solving

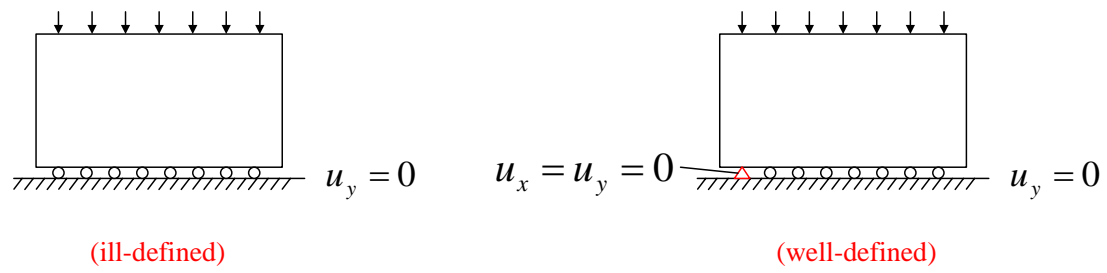
$$\mathbf{K}\mathbf{U} = \mathbf{F}$$

(Can you explain why?)

Example 1:



Example 2:

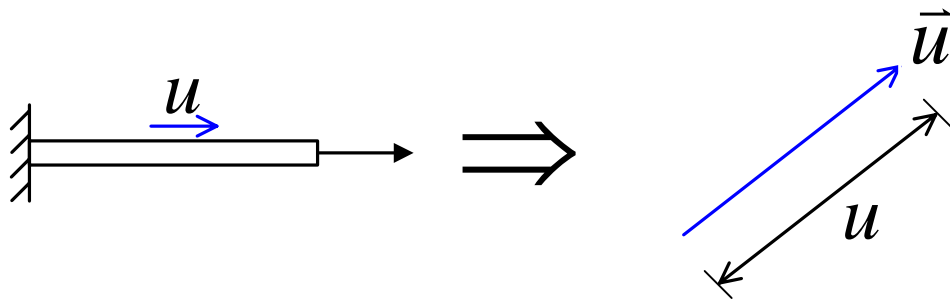


(End of FEM introduction)

2. Mathematical Background

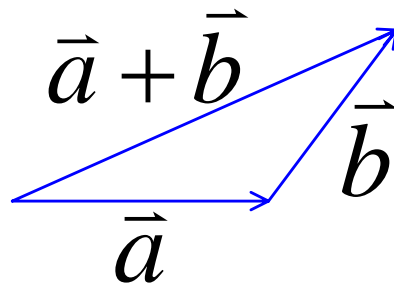
{ Displacements
 { Stresses
 { Strains

1D to 3D:

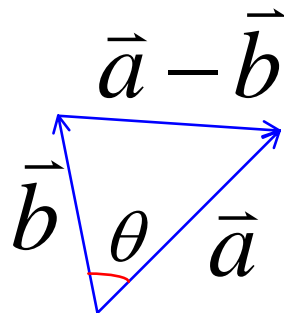


In 3D problems, \vec{u} is a vector (magnitude $|\vec{u}| = u$ plus direction)

Summation of vectors:



Subtraction of vectors:



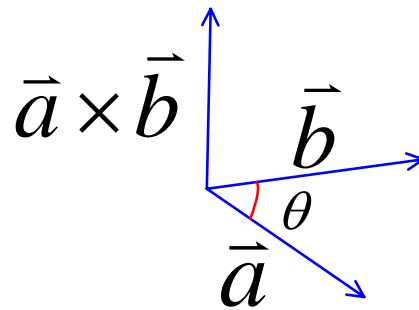
Dot product (or scalar product) of vectors:

$$\vec{a} \cdot \vec{b} = ab \cos \theta$$

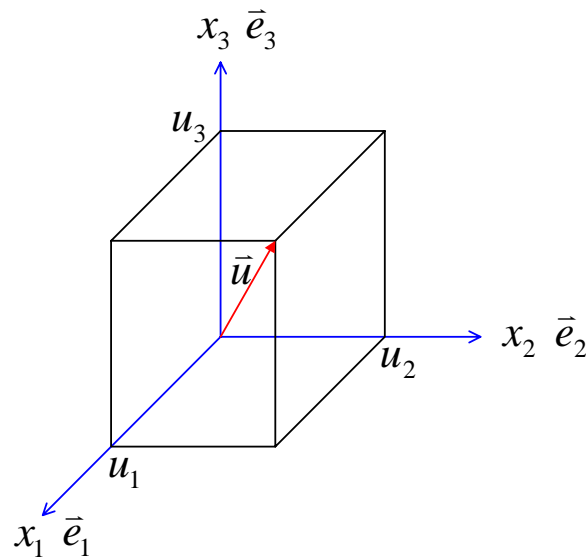
If $\vec{a} \cdot \vec{b} = 0$, \vec{a} is \perp to \vec{b} ($\theta = 90^\circ$)

Cross product (or vector product) of vectors:

$$|\vec{a} \times \vec{b}| = ab \sin \theta$$



To define a vector, we need a coordinate.



\vec{e}_1 , \vec{e}_2 , \vec{e}_3 are defined as base vectors.

$$|\vec{e}_1| = |\vec{e}_2| = |\vec{e}_3| = 1.$$

$$\vec{u} = u_1\vec{e}_1 + u_2\vec{e}_2 + u_3\vec{e}_3 = \sum_{k=1}^3 u_k\vec{e}_k = u_k\vec{e}_k \quad (\text{Einstein Summation Convention})$$

Note that the index k is completely arbitrary. It is just a notation for summation over 1, 2, 3. One could alternatively write

$$\vec{u} = u_k\vec{e}_k = u_i\vec{e}_i = u_j\vec{e}_j = \dots$$

	Index notation	Matrix notation
Vectors:	u_i	$\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$
Matrices:	A_{ij}	$\begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}$

For base vectors $\vec{e}_1, \vec{e}_2, \vec{e}_3$

$$\vec{e}_1 \cdot \vec{e}_1 = 1, \quad \vec{e}_1 \cdot \vec{e}_2 = 0, \quad \vec{e}_1 \cdot \vec{e}_3 = 0$$

$$\vec{e}_2 \cdot \vec{e}_1 = 0, \quad \vec{e}_2 \cdot \vec{e}_2 = 1, \quad \vec{e}_2 \cdot \vec{e}_3 = 0$$

$$\vec{e}_3 \cdot \vec{e}_1 = 0, \quad \vec{e}_3 \cdot \vec{e}_2 = 0, \quad \vec{e}_3 \cdot \vec{e}_3 = 1$$

in more concise form

$$\vec{e}_i \cdot \vec{e}_j = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases} = \delta_{ij}$$

where δ_{ij} is called **Kronecker Delta**.

In matrix notation

$$\delta_{ij} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

which is called the **Identity Matrix** because

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$$

We can also perform the calculation by using index notation:

$$\delta_{ij}u_i = \sum_{i=1}^3 \delta_{ij}u_i = u_j$$

Dot product of $\vec{u} = u_i\vec{e}_i$ and $\vec{v} = v_j\vec{e}_j$:

By index notation:

$$\vec{u} \cdot \vec{v} = (u_i\vec{e}_i) \cdot (v_j\vec{e}_j) = u_iv_j(\vec{e}_i \cdot \vec{e}_j) = u_iv_j\delta_{ij} = u_iv_i = u_1v_1 + u_2v_2 + u_3v_3$$

By matrix notation:

$$\vec{u} \cdot \vec{v} \rightarrow \begin{bmatrix} u_1 & u_2 & u_3 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = u_1v_1 + u_2v_2 + u_3v_3$$

More examples of calculations using index notation:

$$1) \delta_{ii} = \sum_{i=1}^3 \delta_{ii} = \delta_{11} + \delta_{22} + \delta_{33} = 3$$

$$2) \delta_{ij}\delta_{ij} = \delta_{ii} = 3$$

$$3) \delta_{ij}\delta_{jk} = \delta_{ik}$$

$$4) A_{ij}B_{jk} = \sum_{i=1}^3 A_{ij}B_{jk} \text{ is the index notation of } \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix} \begin{bmatrix} B_{11} & B_{12} & B_{13} \\ B_{21} & B_{22} & B_{23} \\ B_{31} & B_{32} & B_{33} \end{bmatrix}.$$