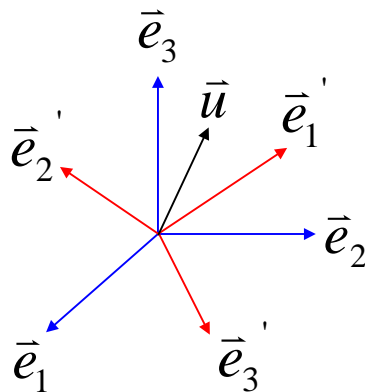


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Review on coordinate transformation (change of basis) for tensors.



$$\bar{u} = u_i \bar{e}_i = u'_p \bar{e}'_p$$

$$u'_p = \bar{u} \cdot \bar{e}'_p = u_i (\bar{e}_i \cdot \bar{e}'_p)$$

$$u_i = \bar{u} \cdot \bar{e}_i = u'_q (\bar{e}'_q \cdot \bar{e}_i)$$

Combining the above 2 equations yields  $u'_p = u'_q (\bar{e}_i \cdot \bar{e}'_q) (\bar{e}'_i \cdot \bar{e}_p) = u'_q Q_{iq} Q_{ip}$

Therefore  $Q_{iq} Q_{ip} \equiv \delta_{pq}$

Similarly, we can show  $Q_{ip} Q_{jp} \equiv \delta_{ij}$

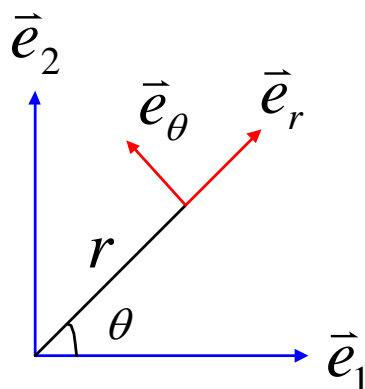
(Hint: using  $u_i = u_j (\bar{e}_j \cdot \bar{e}'_p) (\bar{e}'_i \cdot \bar{e}_p) = u_j Q_{ip} Q_{jp}$ )

In matrix form:

$$\underline{Q} \underline{Q}^T = \underline{I}, \quad \underline{Q}^T \underline{Q} = \underline{I}$$

Such matrices/tensors are called orthogonal matrices/tensors.

Example: Transformation from 2D Cartesian coordinate to 2D Polar coordinate.



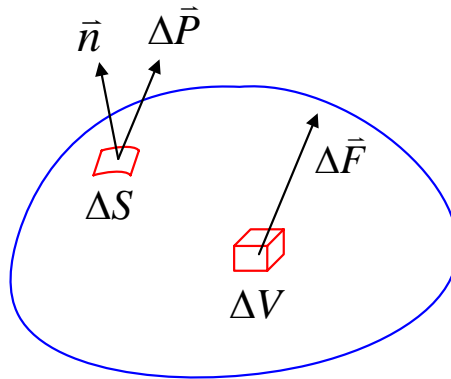
$$\underline{Q} = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}, \quad \underline{Q}\underline{Q}^T = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix} \begin{bmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

### Chap 3. Stress in a solid

**Continuum** — Continuous media ignoring the atomic and other discreteness of matters.

**Density** —  $\rho = \frac{\Delta M}{\Delta V}$  ( $\Delta V \rightarrow 0$ ).

( $\Delta V$  is smaller than all important dimensions but still contains sufficient number of atoms)



**Homogeneity** — All points have the same material properties.  
(Opposite term: Heterogeneity)

**Isotropy** — Material properties are the same in different directions.  
(Opposite term: Anisotropy)

Forces in a continuum:

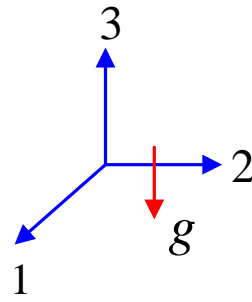
**External (applied) forces**

**Body (volume) forces**

$\Delta \vec{F}$  — total body force on  $\Delta V$  (e.g. gravity).

$$\vec{f} = \lim_{\Delta V \rightarrow 0} \left( \frac{\Delta \vec{F}}{\Delta V} \right)$$

For example: The body force due to gravity can be written as



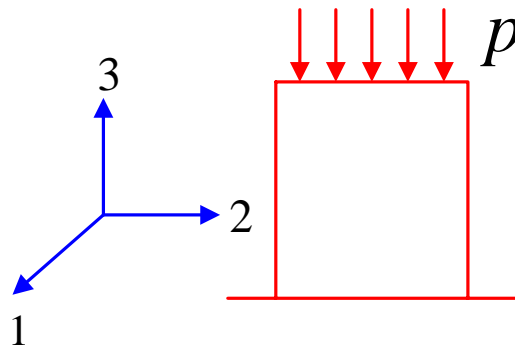
$$\vec{f} = -\rho g \vec{e}_3$$

### Surface forces

$\Delta \vec{P}$  — total surface forces on  $\Delta S$ .

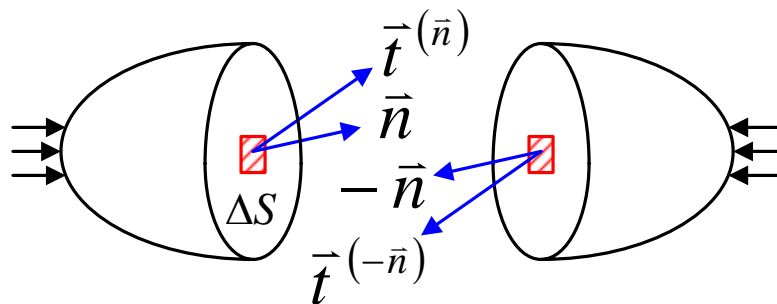
$$\vec{t} = \lim_{\Delta S \rightarrow 0} \left( \frac{\Delta \vec{P}}{\Delta S} \right)$$

For example: A uniform pressure on top of a block can be written as



$$\vec{t} = -p \vec{e}_3$$

### Internal forces

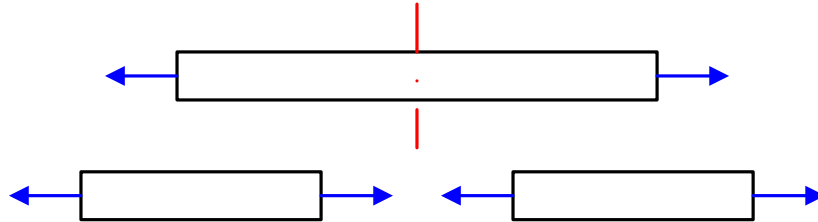


$\vec{t}^{(\vec{n})}$ : traction vector at a point.

According to Newton's action-reaction law,

$$\vec{t}^{(\vec{n})} = -\vec{t}^{(-\vec{n})}$$

Simple example for 1D:



Let us put these concepts in terms of base vectors:

$$\vec{t}^{(\vec{e}_1)} = \bar{t}_1^{(\vec{e}_1)} \vec{e}_1 + \bar{t}_2^{(\vec{e}_1)} \vec{e}_2 + \bar{t}_3^{(\vec{e}_1)} \vec{e}_3$$

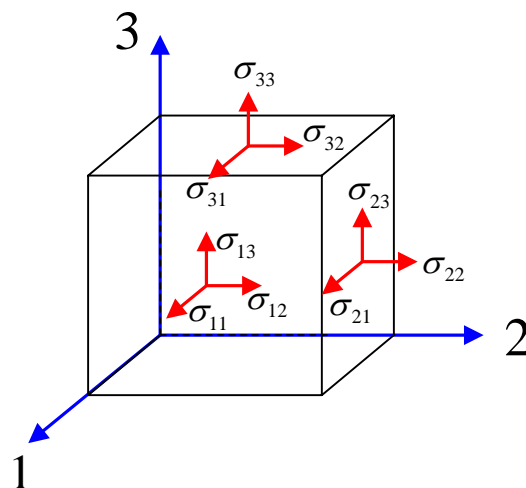
$$\vec{t}^{(\vec{e}_2)} = \bar{t}_1^{(\vec{e}_2)} \vec{e}_1 + \bar{t}_2^{(\vec{e}_2)} \vec{e}_2 + \bar{t}_3^{(\vec{e}_2)} \vec{e}_3$$

$$\vec{t}^{(\vec{e}_3)} = \bar{t}_1^{(\vec{e}_3)} \vec{e}_1 + \bar{t}_2^{(\vec{e}_3)} \vec{e}_2 + \bar{t}_3^{(\vec{e}_3)} \vec{e}_3$$

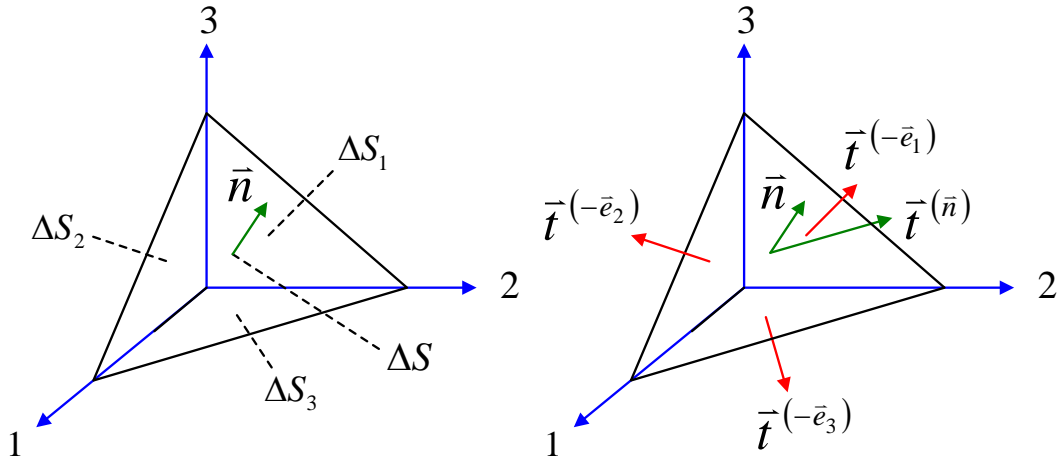
We can define

$$\sigma_{ij} = \bar{t}_j^{(\vec{e}_i)},$$

which is the so-called Cauchy stress. The subscript  $i$  denotes the direction of plane normal and  $j$  denotes the direction of force.



Traction on an arbitrary plane with normal vector  $\vec{n}$



Consider force equilibrium on the tetrahedron shown above:

$$\vec{t}^{(\vec{n})} \Delta S + \vec{t}^{(-\vec{e}_1)} \Delta S_1 + \vec{t}^{(-\vec{e}_2)} \Delta S_2 + \vec{t}^{(-\vec{e}_3)} \Delta S_3 + \vec{f} \Delta V = 0,$$

where  $\vec{f}$  is the body force (which can also include inertia forces). Let  $\Delta S \rightarrow 0$ ,

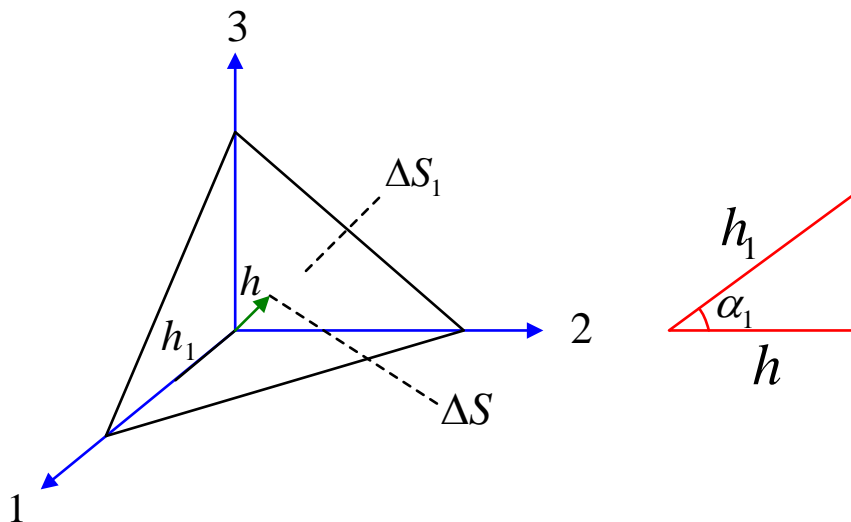
$$\vec{t}^{(\vec{n})} + \vec{t}^{(-\vec{e}_1)} \frac{\Delta S_1}{\Delta S} + \vec{t}^{(-\vec{e}_2)} \frac{\Delta S_2}{\Delta S} + \vec{t}^{(-\vec{e}_3)} \frac{\Delta S_3}{\Delta S} + \vec{f} \frac{\Delta V}{\Delta S} = 0$$

$\frac{\Delta V}{\Delta S} \sim \Delta S^{\frac{1}{2}} \rightarrow 0$  as  $\Delta S \rightarrow 0$  (This is the same as saying that the surface-to-volume ratio

becomes very large as the volume shrinks to zero.)

Consider the volume of the tetrahedron,  $V = \frac{1}{3} \Delta S \cdot h = \frac{1}{3} \Delta S_1 \cdot h_1$

$$\Rightarrow \frac{\Delta S_1}{\Delta S} = \frac{h}{h_1} = \cos \alpha_1 = n \cdot \vec{e}_1 = n_1$$



Similarly,

$$\frac{\Delta S_2}{\Delta S} = n_2, \quad \frac{\Delta S_3}{\Delta S} = n_3$$

where  $n_1$ ,  $n_2$ ,  $n_3$  are the components of the normal vector  $\bar{n}$  ( $\bar{n} = n_1\bar{e}_1 + n_2\bar{e}_2 + n_3\bar{e}_3$ ).

Hence,

$$\bar{t}^{(\bar{n})} = \bar{t}^{(\bar{e}_1)}n_1 + \bar{t}^{(\bar{e}_2)}n_2 + \bar{t}^{(\bar{e}_3)}n_3$$

In index notation,

$$\bar{t}_j^{(\bar{n})} = \bar{t}_j^{(\bar{e}_i)}n_i = \sigma_{ij}n_i$$

i.e.

$$\bar{t}^{(\bar{n})} = \underline{\sigma}^T \bar{n}$$

Special cases:

$$\bar{t}^{(\bar{e}_1)} = \underline{\sigma} \bar{e}_1 = \sigma_{ij}(\bar{e}_i \otimes \bar{e}_j)\bar{e}_i = \sigma_{1j}\bar{e}_j$$

$$\bar{t}^{(\bar{e}_2)} = \dots = \sigma_{2j}\bar{e}_j$$

$$\bar{t}^{(\bar{e}_3)} = \dots = \sigma_{3j}\bar{e}_j$$

$\sigma_{11}$ ,  $\sigma_{22}$ ,  $\sigma_{33}$  are called the normal stresses (on a base plane).

$\sigma_{12}$ ,  $\sigma_{13}$ ,  $\sigma_{23}$  etc. are called the shear stresses (on a base plane).