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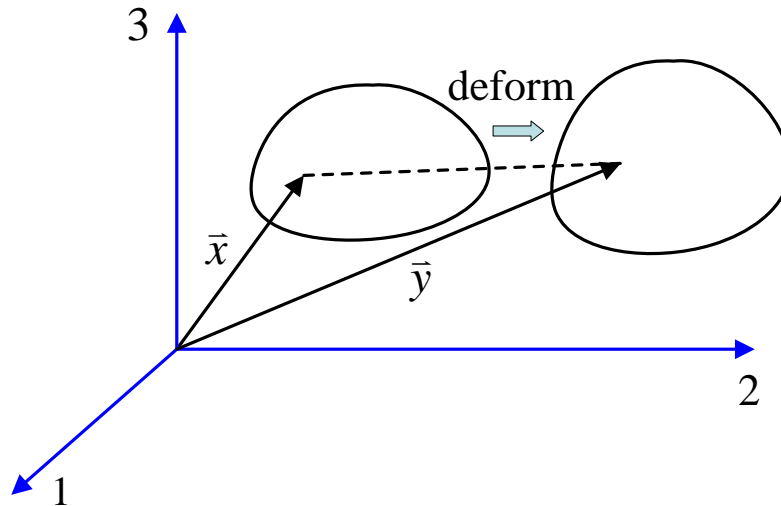
سایت آموزش مهندسی مکانیک ایران

Announcements:

Mid-term (Tuesday, Oct 31) (1 page double sided notes allowed)

Proposal/team for ABAQUS project (Thursday, Nov 9)

### Review of strain concepts



$$\bar{y} = \bar{y}(\bar{x}, t), \quad y_i = y_i(x_1, x_2, x_3, t) = x_i + u_i$$

$$\text{Deformation gradient: } d\bar{y} = \underline{F}d\bar{x}, \quad F_{ij} = \frac{\partial y_i}{\partial x_j}$$

Cauchy-Green strains:

$$\underline{C} = \underline{F}^T \underline{F}, \quad C_{ij} = F_{ki} F_{kj}$$

$$\underline{B} = \underline{F} \underline{F}^T, \quad B_{ij} = F_{ik} F_{jk}$$

$$\text{Stretch tensors: } \underline{U} = \sqrt{\underline{C}}, \quad \underline{V} = \sqrt{\underline{B}}$$

$$\underline{C} = \begin{bmatrix} \lambda_I^2 & 0 & 0 \\ 0 & \lambda_{II}^2 & 0 \\ 0 & 0 & \lambda_{III}^2 \end{bmatrix}, \quad \underline{U} = \begin{bmatrix} \lambda_I & 0 & 0 \\ 0 & \lambda_{II} & 0 \\ 0 & 0 & \lambda_{III} \end{bmatrix} = \sqrt{\underline{C}}$$

Polar decomposition:

$$\underline{F} = \underline{R} \underline{U} = \underline{V} \underline{R}$$

Let us see how the principal values/directions of C-G strains are related:

The principal values/directions of  $\underline{C}$  and  $\underline{U}$  are

$$\underline{U}\bar{m} = \lambda_U \bar{m}, \quad \underline{C}\bar{m} = \lambda_U^2 \bar{m}$$

Multiply the above equation by  $\underline{R}$ ,

$$\underline{RU}\bar{m} = \lambda_U \underline{R}\bar{m}$$

Since  $\underline{RU} = \underline{VR} = \underline{F}$ , we have

$$\underline{V}(\underline{R}\bar{m}) = \lambda_U (\underline{R}\bar{m})$$

Comparing this with the principal value equations for  $\underline{V}$  and  $\underline{B}$ ,

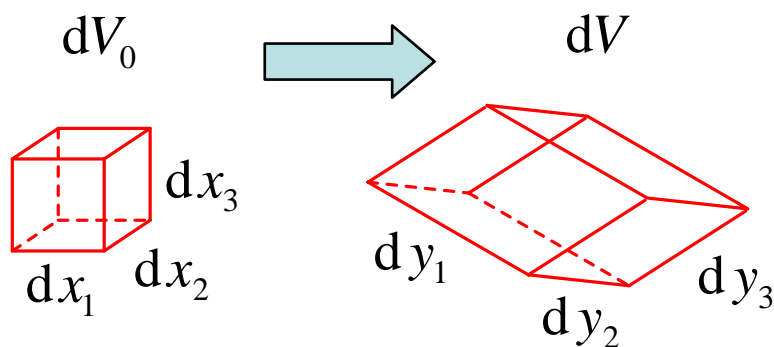
$$\underline{V}\bar{n} = \lambda_V \bar{n} \quad (\text{and} \quad \underline{B}\bar{n} = \lambda_V^2 \bar{n})$$

we can see that

$$\lambda_U = \lambda_V = \lambda, \quad \bar{n} = \underline{R}\bar{m}$$

Therefore, we have one set of principle values  $(\lambda_I, \lambda_{II}, \lambda_{III})$  and two sets of principle directions,  $(\bar{m}_I, \bar{m}_{II}, \bar{m}_{III})$  for  $\underline{C}$ ,  $\underline{U}$  and  $(\bar{n}_I, \bar{n}_{II}, \bar{n}_{III}) = \underline{R}(\bar{m}_I, \bar{m}_{II}, \bar{m}_{III})$  for  $\underline{V}$  and  $\underline{B}$ .

### Volume change



$\frac{dV}{dV_0}$ : ratio of volume change (dilation) during deformation

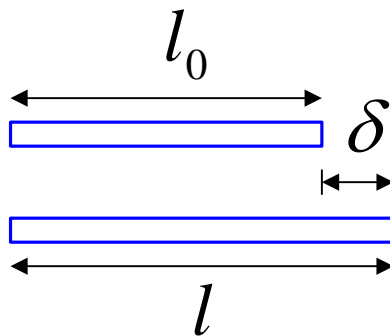
$$\begin{aligned}
 dV &= (dy_1 \times dy_2) \cdot dy_3 = \varepsilon_{ijk} dy_i dy_j dy_k \\
 &= \varepsilon_{ijk} F_{ip} F_{jq} F_{kr} dx_p dx_q dx_r \\
 &= \varepsilon_{pqr} \det(\underline{F}) dx_p dx_q dx_r \\
 &= \det(\underline{F}) dV_0
 \end{aligned}$$

Therefore:  $\frac{dV}{dV_0} = \det(\underline{F}) = \det(\underline{U}) = \det(\underline{V}) = \sqrt{\det(\underline{B})} = \sqrt{\det(\underline{C})} = J$

$J$  is called the Jacobian of deformation.

$$J = \begin{vmatrix} \partial y_1 / \partial x_1 & \partial y_1 / \partial x_2 & \partial y_1 / \partial x_3 \\ \partial y_2 / \partial x_1 & \partial y_2 / \partial x_2 & \partial y_2 / \partial x_3 \\ \partial y_3 / \partial x_1 & \partial y_3 / \partial x_2 & \partial y_3 / \partial x_3 \end{vmatrix}$$

### Engineering concepts of strain



$$\varepsilon_{(1)} = \frac{\delta}{l_0} = \frac{l - l_0}{l_0} = \lambda - 1$$

$$\varepsilon_{(2)} = \frac{\delta}{l} = \frac{l - l_0}{l} = 1 - \lambda^{-1}$$

$$\varepsilon_{(3)} = \frac{l^2 - l_0^2}{2l_0^2} = \frac{1}{2}(\lambda^2 - 1)$$

$$\varepsilon_{(4)} = \frac{l^2 - l_0^2}{2l^2} = \frac{1}{2}(1 - \lambda^{-2})$$

For 3D, we can generalize the corresponding definition as

$$(1): (\underline{U} - \underline{I})$$

$$(2): (\underline{I} - \underline{V})$$

$$(3): \frac{1}{2}(\underline{C} - \underline{I})$$

$$(4): \frac{1}{2}(\underline{I} - \underline{B}^{-1})$$

Among them,

$\underline{E} = \frac{1}{2}(\underline{C} - \underline{I})$  is called the Lagrangian strain tensor.

$\underline{E}^* = \frac{1}{2}(\underline{I} - \underline{B}^{-1})$  is called the Eulerian strain tensor.

Example 1: Pure dilation (no shear)

$$\underline{U} = \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} = \lambda \underline{I}$$

$$\underline{C} = \lambda^2 \underline{I}$$

$$\underline{F} = \underline{R}\underline{U} = \lambda \underline{R}$$

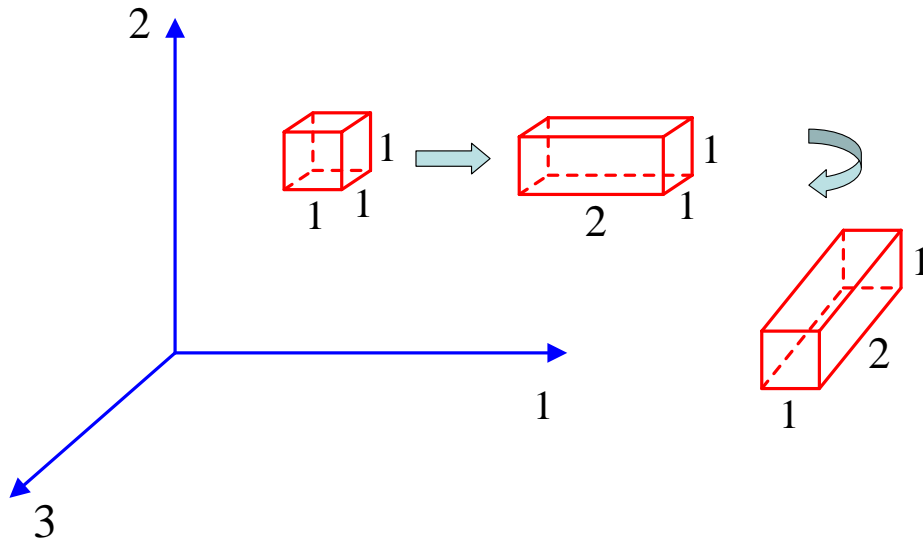
$$\underline{B} = \underline{F}\underline{F}^T = \lambda^2 \underline{I}$$

$$\underline{V} = \lambda \underline{I}$$

$$\underline{E} = \frac{1}{2}(\lambda^2 - 1)\underline{I} = \underline{E}^*$$

$$\frac{dV}{dV_0} = J = \det(\underline{F}) = \lambda^3$$

Example 2: Uniaxial stretch (stretch along 1-direction first, then rotate  $90^\circ$  about 2-axis)



$$\lambda_I = \lambda, \quad \lambda_{II} = \lambda_{III} = 1$$

$$\underline{U} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\underline{R} = \underline{\bar{e}}_i \cdot \underline{R} \underline{\bar{e}}_j = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix}$$

$$\underline{F} = \underline{R}\underline{U} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -2 & 0 & 0 \end{bmatrix}$$

$$\underline{V}\underline{R} = \underline{F} \Rightarrow \underline{V} = \underline{F}\underline{R}^T = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -2 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$$\underline{C} = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \underline{B} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

$$\underline{E} = \frac{1}{2}(\underline{C} - \underline{I}) = \frac{1}{2} \begin{bmatrix} 3 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\underline{E}^* = \frac{1}{2}(\underline{I} - \underline{B}^{-1}) = \frac{1}{2} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

$$\frac{dV}{dV_0} = J = 2$$

If rotated  $45^\circ$  about 2-axis,

$$\underline{U} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

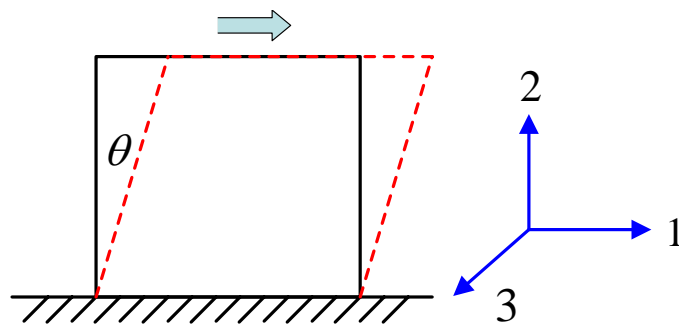
$$\underline{R} = \begin{bmatrix} \cos 45^\circ & -\sin 45^\circ & 0 \\ \sin 45^\circ & \cos 45^\circ & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The other tensors can be worked out by

$$\underline{F} = \underline{R}\underline{U}, \quad \underline{V}\underline{R} = \underline{R}\underline{U}$$

$$\underline{V} = \underline{F}\underline{R}^T, \quad \underline{B} = \underline{F}\underline{F}^T$$

Example 3: Simple shear



$$y_1 = x_1 + x_2 \tan \theta$$

$$y_2 = x_2$$

$$y_3 = x_3$$

$$\underline{F} = \frac{\partial y_i}{\partial x_j} = \begin{bmatrix} 1 & \tan \theta & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\underline{C} = \underline{F}^T \underline{F} = \begin{bmatrix} 1 & 0 & 0 \\ \tan \theta & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & \tan \theta & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & \tan \theta & 0 \\ \tan \theta & 1 + \tan^2 \theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

### Small strain & small rotation theory

Assumption: displacements in solids are usually small compared to relevant structure sizes, i.e.

$$\frac{\partial u_i}{\partial x_j} \propto \left( \frac{u}{L} \right) \ll 1$$

In this case, it proves to be useful to Linearize all equations about  $\frac{\partial u_i}{\partial x_j}$ ,

$$\underline{F} = \frac{\partial y_i}{\partial x_j} = \left( \delta_{ij} + \frac{\partial u_i}{\partial x_j} \right) = \delta_{ij} + u_{i,j} \quad (\text{because } y_i = x_i + u_i)$$

$$\underline{C} = F_{ki} F_{kj} = (\delta_{ki} + u_{k,i})(\delta_{kj} + u_{k,j}) = \delta_{ij} + u_{i,j} + u_{j,i} + u_{k,i} u_{k,j} \cong \delta_{ij} + u_{i,j} + u_{j,i}$$

$$\underline{U} = \sqrt{\underline{C}} = \delta_{ij} + \frac{1}{2}(u_{i,j} + u_{j,i})$$

$$\underline{E} = \frac{1}{2}(\underline{C} - \underline{I}) = \frac{1}{2}(u_{i,j} + u_{j,i}) = \varepsilon_{ij} \quad (\text{small strain tensor})$$

$$\underline{B} = F_{ik} F_{jk} = (\delta_{ik} + u_{i,k})(\delta_{jk} + u_{j,k}) \cong \delta_{ij} + u_{i,j} + u_{j,i}$$

$$\underline{V} = \delta_{ij} + \frac{1}{2}(u_{i,j} + u_{j,i})$$

$$\underline{E}^* = \frac{1}{2}(\underline{I} - \underline{B}^{-1}) = \frac{1}{2}(u_{i,j} + u_{j,i}) = \varepsilon_{ij} = \underline{E}$$

For small strain & small rotation, there is no longer a need to distinguish between Lagrangian and Eulerian strain tensors. Basically, all strain tensors become reduced to one

$$\varepsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i})$$

This greatly simplifies the mathematical problem.