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سایت آموزش مهندسی مکانیک ایران

Mechanical Behavior of Solids

Linear Elastic solids



$$\sigma = E\varepsilon \quad (1D) \Rightarrow \sigma_{ij} = C_{ijkl}\varepsilon_{kl} \quad \text{or} \quad \varepsilon_{ij} = S_{ijkl}\sigma_{kl} \quad (3D)$$

where \underline{C} is sometimes called the stiffness tensor and \underline{S} is sometimes called the compliance tensor. Both of them are 4th order elastic moduli tensors.

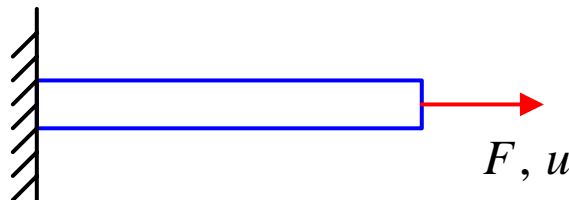
Symmetry of elastic moduli tensors:

$$C_{ijkl} = C_{jikl}, \quad C_{ijkl} = C_{ijlk} \quad (\text{minor symmetry})$$

The minor symmetries reduce the independent elastic constants from 81 to 36.

There is also a major symmetry in elastic moduli $C_{ijkl} = C_{klij}$, which reduces the number of independent elastic constants from 36 to 21.

We use the concept of energy & work to demonstrate the major symmetry of \underline{C} and \underline{S} :



Assume an increment of displacement at the bar end,

$$u \rightarrow u + \delta u$$

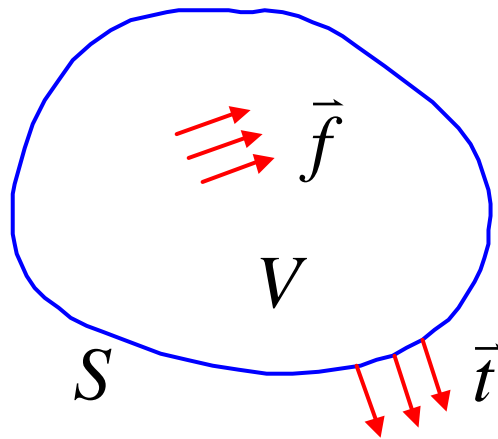
The work done by the applied load should equal to the stored energy in the material,

$$\delta W = F\delta u = \sigma A \delta(l\varepsilon) = Al\sigma\delta\varepsilon = V\sigma\delta\varepsilon$$

$\delta w = \frac{\delta W}{V} = \sigma\delta\varepsilon$ should be the stored elastic energy per unit volume, which is also called the strain energy density.

$$w = w(\varepsilon), \quad \sigma = \frac{\partial w}{\partial \varepsilon}, \quad w(\varepsilon) = \int_0^\varepsilon \sigma \, d\varepsilon$$

Generalize to 3D



$$\begin{aligned}
 \delta w &= \int_S \vec{t} \cdot \delta \vec{u} \, dS + \int_V \vec{f} \cdot \delta \vec{u} \, dV \\
 &= \int_S t_i \delta u_i \, dS + \int_V f_i \delta u_i \, dV \\
 &= \int_S \sigma_{ij} n_j \delta u_i \, dS + \int_V f_i \delta u_i \, dV \\
 &= \int_V (\sigma_{ij} \delta u_i)_{,j} \, dV + \int_V f_i \delta u_i \, dV \\
 &= \int_V (\sigma_{ij,j} + f_i) \delta u_i \, dV + \int_V \sigma_{ij} \delta u_{i,j} \, dV \\
 &= \int_V \sigma_{ij} \delta \varepsilon_{ij} \, dV = \int_V \delta w \, dV
 \end{aligned}$$

$\delta w = \sigma_{ij} \delta \varepsilon_{ij}$ is the strain energy density and

$w = \int \sigma_{ij} \delta \varepsilon_{ij}$ must be integrable.

Knowing $\sigma_{ij} = C_{ijkl} \varepsilon_{kl}$, we have

$$\begin{aligned}
 w &= \frac{1}{2} C_{ijkl} \varepsilon_{ij} \varepsilon_{kl} \\
 C_{ijkl} &= \frac{\partial^2 w}{\partial \varepsilon_{ij} \partial \varepsilon_{kl}} = C_{klij}
 \end{aligned}$$

Elastic moduli for isotropic materials:

Isotropic tensor: $\underline{I} \vec{a} = \vec{a}$ ($\delta_{ij} a_j = a_i$)

The 4th order isotropic tensors can be generally written as

$$C_{ijkl} = c_1 \delta_{ij} \delta_{kl} + c_2 \delta_{ik} \delta_{jl} + c_3 \delta_{il} \delta_{jk}$$

Imposing the symmetry conditions indicate that the elastic moduli can be expressed in terms of just two constants

$$C_{ijkl} = \lambda \delta_{ij} \delta_{kl} + \mu (\delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk})$$

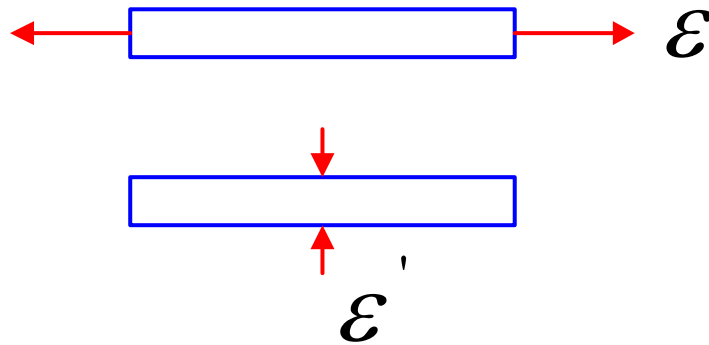
where λ and μ are called Lamé constants.

Similarly, the S_{ijkl} can also be expressed in terms of just two constants as

$$S_{ijkl} = d_1 \delta_{ij} \delta_{kl} + d_2 (\delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk})$$

The constants λ , μ , d_1 and d_2 are to be determined experiments.

Since there are only 2 independent elastic constants, it suffices to conduct experiments in 1D:



$$E = \frac{\sigma}{\varepsilon} \quad (\text{Young's modulus})$$

$$\nu = -\frac{\varepsilon'}{\varepsilon} \quad (\text{Poisson's ratio})$$

:

We can specify the general stress-strain relation

$$\varepsilon_{ij} = S_{ijkl} \sigma_{kl} = (d_1 \delta_{ij} \delta_{kl} + d_2 (\delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk})) \sigma_{kl} = d_1 \sigma_{kk} \delta_{ij} + 2d_2 \sigma_{ij}$$

to 1D loading $\sigma_{11} \neq 0$, $\sigma_{22} = \sigma_{33} = 0$. In this case,

$$\varepsilon_{11} = d_1 \sigma_{11} + 2d_2 \sigma_{11} = (d_1 + 2d_2) \sigma_{11} = \frac{\sigma_{11}}{E}$$

$$\varepsilon_{22} = d_1 \sigma_{11} = -\nu \varepsilon_{11} = -\nu \frac{\sigma_{11}}{E}$$

Therefore,

$$d_1 = -\frac{\nu}{E}$$

$$d_2 = \frac{1+\nu}{2E}$$

and

$$S_{ijkl} = -\frac{\nu}{E} \delta_{ij} \delta_{kl} + \frac{1+\nu}{2E} (\delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk})$$

We thus obtain the so-called generalized Hooke's law:

$$\varepsilon_{ij} = \frac{1+\nu}{E} \sigma_{ij} - \frac{\nu}{E} \sigma_{kk} \delta_{ij}$$

This equation can be inverted as follows.

$$\varepsilon_{qq} = \frac{1+\nu}{E} \sigma_{qq} - \frac{3\nu}{E} \sigma_{kk} = \frac{1-2\nu}{E} \sigma_{qq}$$

The percentage change of volume,

$$\frac{\Delta V}{V} = \varepsilon_{kk} = \frac{1-2\nu}{E} \sigma_{kk} = \frac{3(1-2\nu)}{E} p$$

Defines the so-called bulk modulus $K = \frac{p}{\Delta V/V} = \frac{E}{3(1-2\nu)}$. (This modulus can also be calculated using ab initio quantum mechanics techniques.)

Invert the generalized Hooke's law:

$$\frac{1+\nu}{E} \sigma_{ij} = \varepsilon_{ij} + \frac{\nu}{E} \sigma_{kk} \delta_{ij} = \varepsilon_{ij} + \frac{\nu}{E} \frac{E}{1-2\nu} \varepsilon_{kk} \delta_{ij}$$

$$\Rightarrow \sigma_{ij} = \frac{E}{1+\nu} \varepsilon_{ij} + \frac{\nu E}{(1+\nu)(1-2\nu)} \varepsilon_{kk} \delta_{ij}$$

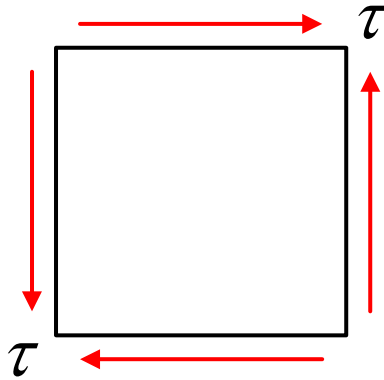
Comparing this to the Lamé form of Hooke's law

$\sigma_{ij} = C_{ijkl} \varepsilon_{kl} = (\lambda \delta_{ij} \delta_{kl} + \mu (\delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk})) \varepsilon_{kl} = \lambda \delta_{ij} \varepsilon_{kk} + 2\mu \varepsilon_{ij}$, we identify

$$\lambda = \frac{\nu E}{(1+\nu)(1-2\nu)}$$

$$\mu = \frac{E}{2(1+\nu)}$$

For shear deformation:



$$\gamma = \frac{\tau}{G} \Rightarrow G = \frac{\tau}{\gamma}$$

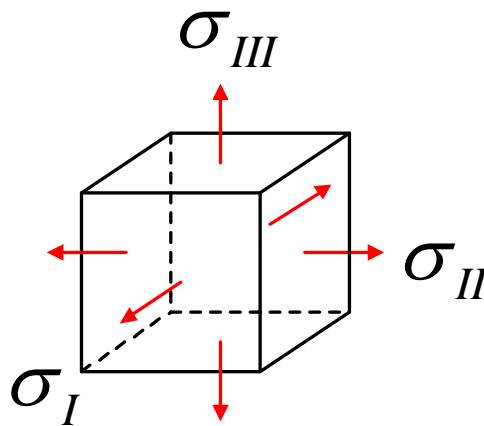
Let $i=1$, $j=2$ in $\sigma_{ij} = \lambda \delta_{ij} \varepsilon_{kk} + 2\mu \varepsilon_{ij}$,

$$\sigma_{12} = \tau = 2\mu \varepsilon_{12} = \mu \gamma$$

Therefore, the second Lamé constant is just the shear modulus:

$$G = \mu = \frac{E}{2(1+\nu)}$$

Alternative derivation of the generalized Hooke's law



Consider a linear superposition of strain in the principal stress directions

$$\varepsilon_I = \frac{\sigma_I}{E} - \nu \frac{\sigma_{II}}{E} - \nu \frac{\sigma_{III}}{E} = \frac{1+\nu}{E} \sigma_I - \frac{\nu}{E} (\sigma_I + \sigma_{II} + \sigma_{III})$$

$$\varepsilon_{II} = \frac{\sigma_{II}}{E} - \nu \frac{\sigma_I}{E} - \nu \frac{\sigma_{III}}{E} = \frac{1+\nu}{E} \sigma_{II} - \frac{\nu}{E} (\sigma_I + \sigma_{II} + \sigma_{III})$$

$$\varepsilon_{III} = \frac{\sigma_{III}}{E} - \nu \frac{\sigma_I}{E} - \nu \frac{\sigma_{II}}{E} = \frac{1+\nu}{E} \sigma_{III} - \frac{\nu}{E} (\sigma_I + \sigma_{II} + \sigma_{III})$$

i.e.

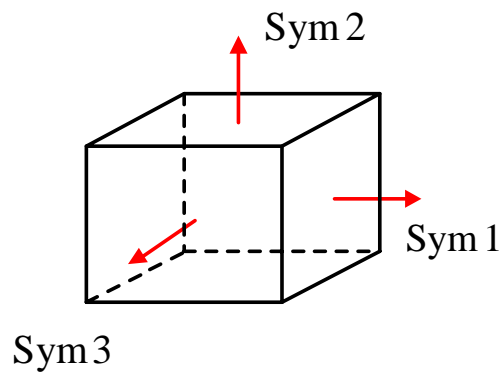
Therefore, we have

$$\underline{\varepsilon} = \frac{1+\nu}{E} \underline{\sigma} - \frac{\nu}{E} \sigma_{kk} \underline{I}$$

Through change of coordinates, this relation remains true for any other coordinate systems.

Orthotropic materials

An orthotropic material (e.g. wood, composites, fiber glass, etc) has three mutually perpendicular symmetry planes.



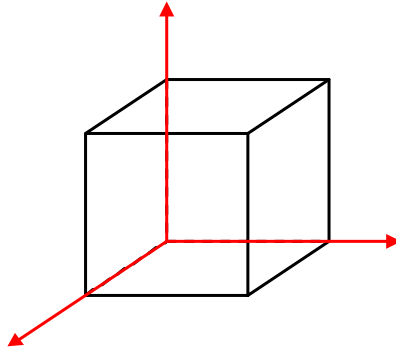
$$\begin{pmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{13} \\ \sigma_{12} \end{pmatrix} = \begin{pmatrix} & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \end{pmatrix} \begin{pmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ \varepsilon_{23} \\ \varepsilon_{13} \\ \varepsilon_{12} \end{pmatrix}$$

$\mathbf{C}_{6 \times 6}$

$$\mathbf{C}: \begin{pmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & 0 \\ & C_{22} & C_{23} & 0 & 0 & 0 \\ & & C_{33} & 0 & 0 & 0 \\ & & & C_{44} & 0 & 0 \\ & Sym & & & C_{55} & 0 \\ & & & & & C_{66} \end{pmatrix} \quad (9 \text{ elastic constants})$$

Cubic materials

Single crystal metals: FCC (Cu, Al, Ag, etc.); BCC (Fe, etc.)



$$\mathbf{C}: \begin{pmatrix} C_{11} & C_{12} & C_{12} & 0 & 0 & 0 \\ & C_{11} & C_{12} & 0 & 0 & 0 \\ & & C_{11} & 0 & 0 & 0 \\ & & & C_{44} & 0 & 0 \\ \text{Sym} & & & & C_{44} & 0 \\ & & & & & C_{44} \end{pmatrix} \quad (3 \text{ elastic constants})$$