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Review of deformation tensors:

$$\underline{F}, \underline{C}, \underline{B}, \underline{U}, \underline{V}, \underline{R}, \underline{E}, \underline{E}^*$$

Given \underline{F} , one can follow the following standard procedure to determine the other strain measures.

$$1) \text{ Most simply, } \underline{C} = \underline{F}^T \underline{F}, \underline{B} = \underline{F} \underline{F}^T, \underline{E} = \frac{1}{2}(\underline{C} - \underline{I}), \underline{E}^* = \frac{1}{2}(\underline{I} - \underline{B}^{-1})$$

2) To find \underline{U} , \underline{V} , \underline{R} , we need to perform the eigenvalue analysis of \underline{C} and \underline{B} (diagonalization of matrices):

$$\underline{C}\bar{m} = \lambda^2 \bar{m} \Rightarrow \underline{C} = \begin{pmatrix} \lambda_I^2 & 0 & 0 \\ 0 & \lambda_{II}^2 & 0 \\ 0 & 0 & \lambda_{III}^2 \end{pmatrix} = \lambda_I^2 \bar{m}_I \otimes \bar{m}_I + \lambda_{II}^2 \bar{m}_{II} \otimes \bar{m}_{II} + \lambda_{III}^2 \bar{m}_{III} \otimes \bar{m}_{III}$$

$$\underline{B}\bar{n} = \lambda^2 \bar{n} \Rightarrow \underline{B} = \begin{pmatrix} \lambda_I^2 & 0 & 0 \\ 0 & \lambda_{II}^2 & 0 \\ 0 & 0 & \lambda_{III}^2 \end{pmatrix} = \lambda_I^2 \bar{n}_I \otimes \bar{n}_I + \lambda_{II}^2 \bar{n}_{II} \otimes \bar{n}_{II} + \lambda_{III}^2 \bar{n}_{III} \otimes \bar{n}_{III}$$

3)

$$\underline{U} = \lambda_I \bar{m}_I \otimes \bar{m}_I + \lambda_{II} \bar{m}_{II} \otimes \bar{m}_{II} + \lambda_{III} \bar{m}_{III} \otimes \bar{m}_{III}$$

$$\underline{V} = \lambda_I \bar{n}_I \otimes \bar{n}_I + \lambda_{II} \bar{n}_{II} \otimes \bar{n}_{II} + \lambda_{III} \bar{n}_{III} \otimes \bar{n}_{III}$$

$$U_{ij} = \bar{e}_i \cdot \underline{U} \bar{e}_j, V_{ij} = \bar{e}_i \cdot \underline{V} \bar{e}_j \Rightarrow \underline{U} \text{ and } \underline{V}$$

4)

$$\underline{F} = \underline{R}\underline{U} = \underline{V}\underline{R}$$

$$\underline{R} = \underline{F}\underline{U}^{-1} = \underline{V}^{-1}\underline{F}$$

Generalized Hooke's law

$$\sigma_{ij} = 2\mu\varepsilon_{ij} + \lambda\varepsilon_{kk}\delta_{ij}$$

$$\varepsilon_{ij} = \frac{1+\nu}{E}\sigma_{ij} - \frac{\nu}{E}\sigma_{kk}\delta_{ij}$$

where

$$\lambda = \frac{\nu E}{(1+\nu)(1-2\nu)}, \mu = \frac{E}{2(1+\nu)}$$

Alternative forms in terms of deviatoric stresses/strains:

$$\sigma_{ij} = \sigma'_{ij} + \frac{\sigma_{kk}}{3} \delta_{ij}$$

$$\varepsilon_{ij} = \varepsilon'_{ij} + \frac{\varepsilon_{kk}}{3} \delta_{ij}$$

$$\varepsilon'_{ij} + \frac{\varepsilon_{kk}}{3} \delta_{ij} = \frac{1+\nu}{E} \left(\sigma'_{ij} + \frac{\sigma_{kk}}{3} \delta_{ij} \right) - \frac{\nu}{E} \sigma_{kk} \delta_{ij} = \frac{1+\nu}{E} \sigma'_{ij} + \frac{1-2\nu}{3E} \sigma_{kk} \delta_{ij}$$

$$\varepsilon_{kk} = \frac{1-2\nu}{E} \sigma_{kk} = \frac{\sigma_{kk}}{3K}$$

$$\varepsilon'_{ij} = \frac{1+\nu}{E} \sigma'_{ij} = \frac{\sigma'_{ij}}{2\mu}$$

$$\varepsilon_{ij} = \varepsilon'_{ij} + \frac{\varepsilon_{kk}}{3} \delta_{ij} = \frac{\sigma'_{ij}}{2\mu} + \frac{\sigma_{kk}}{9K} \delta_{ij}$$

Strain energy:

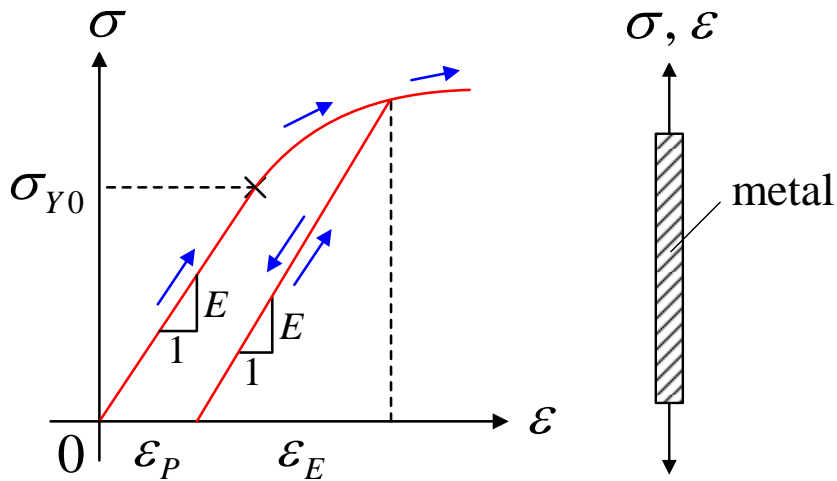
$$dw = \sigma_{ij} d\varepsilon_{ij}$$

For linear elastic solids,

$$\begin{aligned} w &= \int \sigma_{ij} d\varepsilon_{ij} = \frac{1}{2} C_{ijkl} \varepsilon_{ij} \varepsilon_{kl} = \frac{1}{2} \sigma_{ij} \varepsilon_{ij} \\ &= \frac{1}{2} \left(\sigma'_{ij} + \frac{\sigma_{kk}}{3} \delta_{ij} \right) \left(\varepsilon'_{ij} + \frac{\varepsilon_{kk}}{3} \delta_{ij} \right) \\ &= \frac{1}{2} \sigma'_{ij} \varepsilon'_{ij} + \frac{1}{6} \sigma_{kk} \varepsilon_{kk} \end{aligned}$$

$$w = \mu \varepsilon'_{ij} \varepsilon'_{ij} + \frac{1}{2} K \varepsilon_{kk}^2 \quad \text{or} \quad w = \frac{1}{4\mu} \sigma'_{ij} \sigma'_{ij} + \frac{1}{18K} \sigma_{kk}^2$$

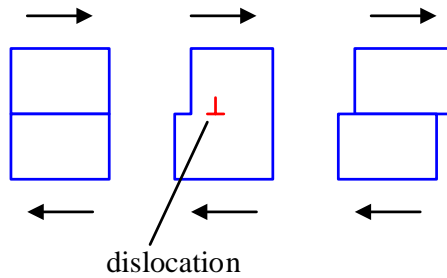
Plastic material behavior



Important experimental facts:

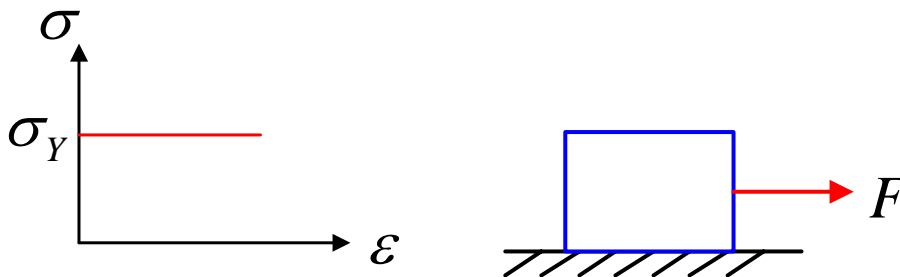
- 1) Hydrostatic stress has no effects on plastic deformation;
- 2) Plastic behavior doesn't induce volume change of a material.

In 1930's, Taylor and other scientists discovered that plastic deformation is caused by shearing of atomic planes via propagation of a type of lattice defects called dislocations.

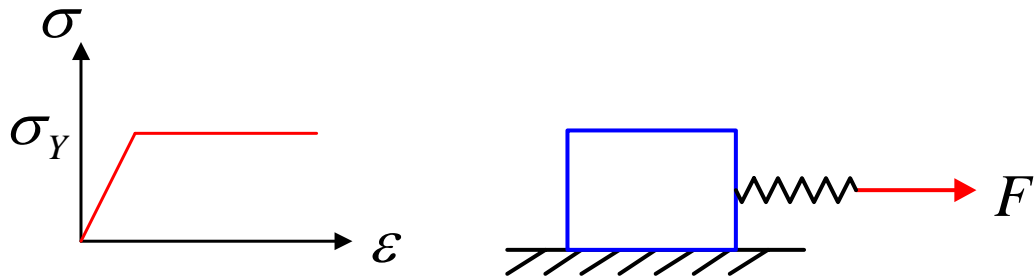


1D models:

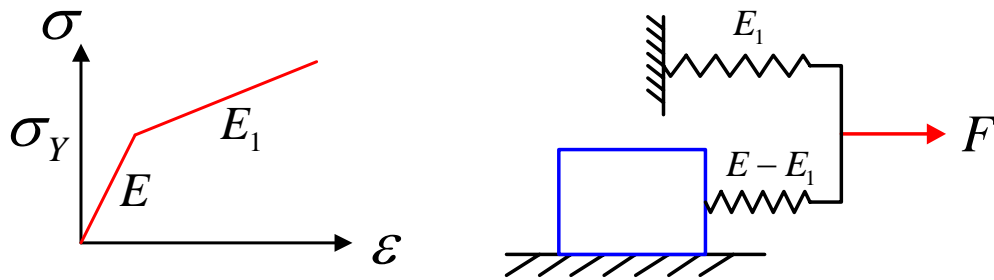
Rigid-Perfectly plastic material



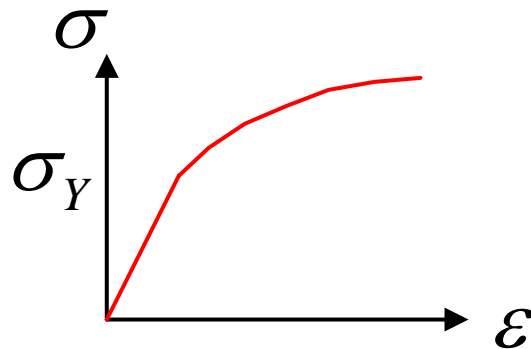
Elastic-perfectly plastic material



Linear hardening material



Power law hardening material



$$\sigma = \begin{cases} E\varepsilon, & \sigma < \sigma_Y \\ \sigma_Y \left(1 + \frac{E\varepsilon_p}{\sigma_Y} \right)^N, & \sigma \geq \sigma_Y \end{cases}$$

Here σ_Y and N can be treated as fitting parameters to experimental data; $0 \leq N < 1$ is called the hardening index.

In incremental form:

$$d\varepsilon_p = \begin{cases} \frac{d\sigma}{h}, & \text{plastic loading} \\ 0, & \text{otherwise} \end{cases}$$

Here $h = \frac{d\sigma}{d\varepsilon_p} = EN \left(1 + \frac{E\varepsilon_p}{\sigma_Y} \right)^{N-1}$ is the tangent modulus of the curve of stress versus plastic strain.

The total strain increment can be decomposed as an elastic and a plastic part as

$$d\varepsilon = d\varepsilon_E + d\varepsilon_P = \frac{d\sigma}{E} + d\varepsilon_P$$

How to generalize to 3D?

- 1) Yield criterion
- 2) Plastic constitutive law ($\sigma - \varepsilon$ relation)

Yield criterion

For 1D,

$$\sigma = \sigma_Y$$

For 3D,

Tresca condition: maximum shear stress = critical \Rightarrow Yield

If $\sigma_I \geq \sigma_{II} \geq \sigma_{III}$,

$$\frac{\sigma_I - \sigma_{III}}{2} = C_Y = \frac{\sigma_Y}{2} \Rightarrow \sigma_I - \sigma_{III} = \sigma_Y$$

For general $\sigma_I, \sigma_{II}, \sigma_{III}$,

$$\text{Max}(|\sigma_I - \sigma_{II}|, |\sigma_I - \sigma_{III}|, |\sigma_{II} - \sigma_{III}|) = \sigma_Y$$

Von Mises condition:

$$w = \frac{1}{4\mu} \sigma'_{ij} \sigma'_{ij} + \frac{1}{18K} \sigma_{kk}^2$$

If we take the distortional part of elastic energy as a criterion for the onset of plastic deformation, we can write the yield condition as

$$\sigma'_{ij} \sigma'_{ij} = C_Y$$

In principal stress orientations,

$$\sigma'_{ij} \sigma'_{ij} = \sigma_I'^2 + \sigma_{II}'^2 + \sigma_{III}'^2$$

$$\sigma'_I = \sigma_I - \frac{1}{3}(\sigma_I + \sigma_{II} + \sigma_{III}) = \frac{1}{3}(2\sigma_I - \sigma_{II} - \sigma_{III})$$

Similarly,

$$\sigma'_{II} = \frac{1}{3}(2\sigma_{II} - \sigma_I - \sigma_{III})$$

$$\sigma'_{III} = \frac{1}{3}(2\sigma_{III} - \sigma_I - \sigma_{II})$$

$$\begin{aligned} \sigma'_{ij}\sigma'_{ij} &= \frac{1}{9} \left((2\sigma_I - \sigma_{II} - \sigma_{III})^2 + (2\sigma_{II} - \sigma_I - \sigma_{III})^2 + (2\sigma_{III} - \sigma_I - \sigma_{II})^2 \right) \\ &= \frac{2}{3} (\sigma_I^2 + \sigma_{II}^2 + \sigma_{III}^2 - \sigma_I\sigma_{II} - \sigma_{II}\sigma_{III} - \sigma_I\sigma_{III}) \end{aligned}$$

Specify to 1D,

$$\frac{2}{3}\sigma^2 = C_Y = \frac{2}{3}\sigma_Y^2 \Rightarrow \sigma'_{ij}\sigma'_{ij} = \frac{2}{3}\sigma_Y^2$$

i.e.

$$\sigma_e = \sqrt{\frac{3}{2}\sigma'_{ij}\sigma'_{ij}} = \sigma_Y$$

which is called the von Mises stress.