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**Summary of elementary strain concepts and their generalizations to 3D tensors:**

<b>1D</b>	<b>3D</b>
$\varepsilon_{(1)} = \frac{\delta}{l_0} = \frac{l-l_0}{l_0} = \lambda - 1$	$(\underline{U} - \underline{I})$
$\varepsilon_{(2)} = \frac{\delta}{l} = \frac{l-l_0}{l} = 1 - \lambda^{-1}$	$(\underline{I} - \underline{V}^{-1})$
$\varepsilon_{(3)} = \frac{l^2 - l_0^2}{2l_0^2} = \frac{1}{2}(\lambda^2 - 1)$	$\frac{1}{2}(\underline{C} - \underline{I})$
$\varepsilon_{(4)} = \frac{l^2 - l_0^2}{2l^2} = \frac{1}{2}(1 - \lambda^{-2})$	$\frac{1}{2}(\underline{I} - \underline{B}^{-1})$
For small strain: $\varepsilon_{(1)} = \varepsilon_{(2)} = \varepsilon_{(3)} = \varepsilon_{(4)} = \varepsilon$	$\underline{\varepsilon} = \underline{U} - \underline{I} = \underline{I} - \underline{V}^{-1} = \frac{1}{2}(\underline{C} - \underline{I}) = \frac{1}{2}(\underline{I} - \underline{B}^{-1})$ $\varepsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i})$ or $\underline{\varepsilon} = \frac{1}{2}(\nabla \bar{u} + \nabla^T \bar{u})$
1 equation: $\varepsilon = \frac{\partial u}{\partial x}$	6 equations: $\varepsilon_{11} = \frac{\partial u_1}{\partial x_1}, \varepsilon_{22} = \frac{\partial u_2}{\partial x_2}, \varepsilon_{33} = \frac{\partial u_3}{\partial x_3}$ $\varepsilon_{12} = \frac{1}{2} \left( \frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1} \right), \varepsilon_{13} = \frac{1}{2} \left( \frac{\partial u_1}{\partial x_3} + \frac{\partial u_3}{\partial x_1} \right),$ $\varepsilon_{23} = \frac{1}{2} \left( \frac{\partial u_2}{\partial x_3} + \frac{\partial u_3}{\partial x_2} \right)$

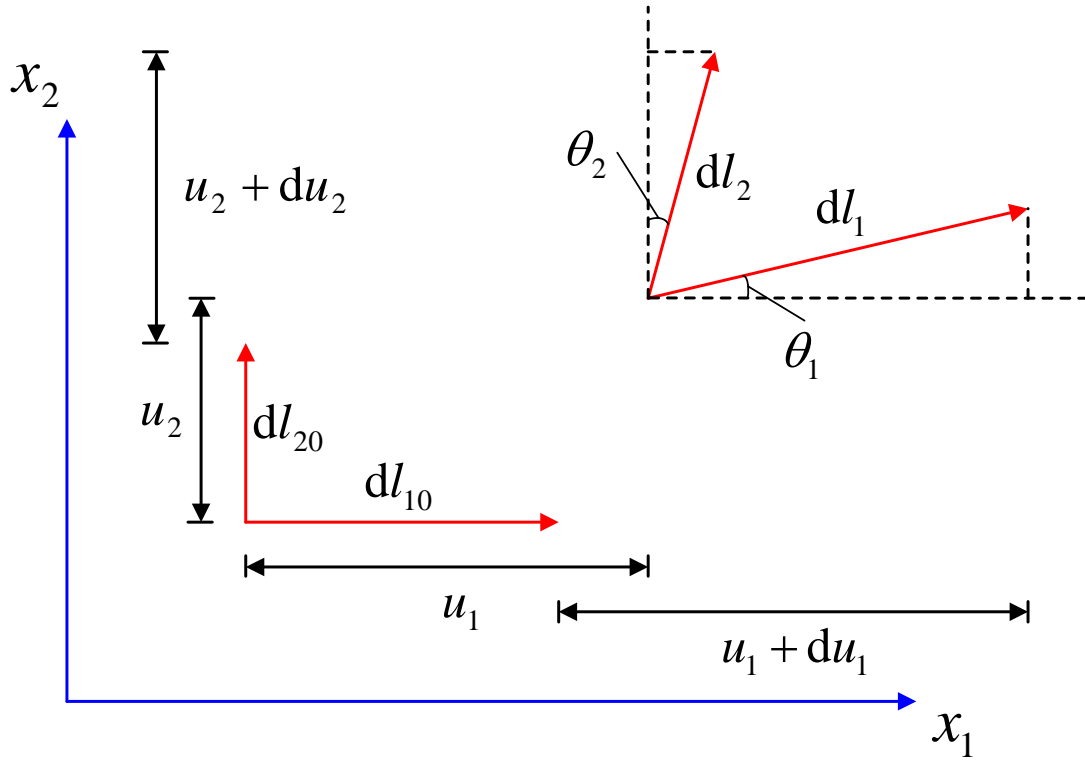
For small strain and small rotation, the rigid body rotation part is analyzed as follows:

$\underline{R} = \underline{I} + \underline{\omega}$  where  $\underline{\omega}$  is defined as the small rotation tensor.

$$\underline{RU} = (\underline{I} + \underline{\omega})(\underline{I} + \underline{\varepsilon}) \cong \underline{I} + \underline{\varepsilon} + \underline{\omega} = \underline{F} = \underline{I} + \nabla \bar{u}$$

$$\underline{\omega} = \nabla \bar{u} - \underline{\varepsilon} = \nabla \bar{u} - \frac{1}{2}(\nabla \bar{u} + \nabla^T \bar{u}) = \frac{1}{2}(\nabla \bar{u} - \nabla^T \bar{u})$$

$$\underline{\omega}^T = -\underline{\omega} \text{ (antisymmetric tensor)}$$

**Alternative way of deriving small strain tensor:**

Normal strain:

$$\varepsilon_{11} = \frac{dl_1 - dl_{10}}{dl_{10}} = \frac{\sqrt{(dl_{10} + du_1)^2 + du_2^2} - dl_{10}}{dl_{10}} = \sqrt{\left(1 + \frac{\partial u_1}{\partial x_1}\right)^2 + \left(\frac{\partial u_2}{\partial x_1}\right)^2} - 1 \cong \frac{\partial u_1}{\partial x_1}$$

$$\varepsilon_{22} = \frac{dl_2 - dl_{20}}{dl_{20}} = \frac{\sqrt{(dl_{20} + du_2)^2 + du_1^2} - dl_{20}}{dl_{20}} = \sqrt{\left(1 + \frac{\partial u_2}{\partial x_2}\right)^2 + \left(\frac{\partial u_1}{\partial x_2}\right)^2} - 1 \cong \frac{\partial u_2}{\partial x_2}$$

Shear strain:

$$\gamma_{12} = \theta_1 + \theta_2$$

$$\tan \theta_1 = \frac{du_2}{dl_{10} + du_1} = \frac{\partial u_2 / \partial x_1}{1 + \partial u_1 / \partial x_1} = \frac{\partial u_2}{\partial x_1} = \theta_1$$

$$\tan \theta_2 = \frac{du_1}{dl_{20} + du_2} = \frac{\partial u_1 / \partial x_2}{1 + \partial u_2 / \partial x_2} = \frac{\partial u_1}{\partial x_2} = \theta_2$$

$$\gamma_{12} = \theta_1 + \theta_2 = \frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1} = 2\varepsilon_{12}$$

Volume change:

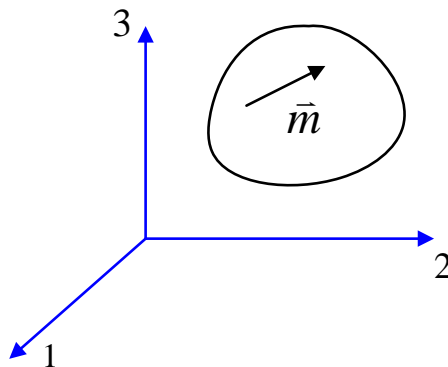
$$\frac{dV}{dV_0} = \det(\underline{F}) = \begin{vmatrix} 1 + \frac{\partial u_1}{\partial x_1} & \frac{\partial u_1}{\partial x_2} & \frac{\partial u_1}{\partial x_3} \\ \frac{\partial u_2}{\partial x_1} & 1 + \frac{\partial u_2}{\partial x_2} & \frac{\partial u_2}{\partial x_3} \\ \frac{\partial u_3}{\partial x_1} & \frac{\partial u_3}{\partial x_2} & 1 + \frac{\partial u_3}{\partial x_3} \end{vmatrix}$$

If  $\left| \frac{\partial u_i}{\partial x_j} \right| \ll 1$ ,

$$\frac{dV}{dV_0} = 1 + \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3} + o\left(\frac{\partial u_i}{\partial x_j}\right) \cong 1 + \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3} = 1 + \varepsilon_{kk}$$

Therefore:

$$\frac{dV - dV_0}{dV_0} = \varepsilon_{kk} \quad (\text{bulk strain})$$

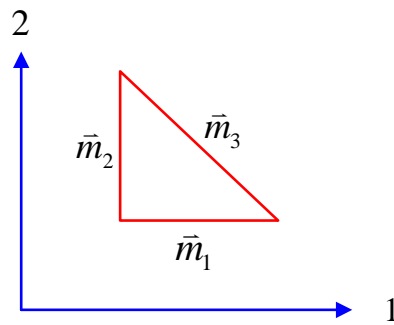


$$\bar{m} \cdot \underline{C} \bar{m} = \lambda^2 \Rightarrow \bar{m} \cdot \frac{1}{2}(\underline{C} - \underline{I}) \bar{m} = \frac{1}{2}(\lambda^2 - 1)$$

Note that  $\underline{\varepsilon} = \frac{1}{2}(\underline{C} - \underline{I})$ ,

$$\bar{m} \cdot \underline{\varepsilon} \bar{m} = \frac{1}{2}(\lambda^2 - 1) \cong \lambda - 1 \quad (\text{strain in } \bar{m} \text{ direction})$$

Strain gages aligned along different directions at a solid surface can be used to measure strain in the plane of the surface. For example, assuming we can measure the normal strain



along the  $\bar{m}_1, \bar{m}_2, \bar{m}_3$  directions as shown above, respectively, the components of surface strain tensor is simply

$$\bar{m}_1 \cdot \underline{\underline{\varepsilon}} \bar{m}_1 = \varepsilon_{11}, \quad \bar{m}_2 \cdot \underline{\underline{\varepsilon}} \bar{m}_2 = \varepsilon_{22}$$

The shear strain  $\varepsilon_{12}$  is to be determined from the reading in the  $\bar{m}_3$  direction.

### Compatibility conditions of strains

3 displacement components  $\Rightarrow$  6 strain components

$$\varepsilon_{ij} = \frac{1}{2} (u_{i,j} + u_{j,i})$$

Eliminating  $u$  from the above  $\Rightarrow$  3 equations among strain (compatibility conditions)

For example:

$$\varepsilon_{11} = \frac{\partial u_1}{\partial x_1}, \quad \varepsilon_{12} = \frac{1}{2} \left( \frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1} \right), \quad \varepsilon_{22} = \frac{\partial u_2}{\partial x_2}$$

$$\frac{\partial^2 \varepsilon_{11}}{\partial x_2^2} + \frac{\partial^2 \varepsilon_{22}}{\partial x_1^2} = \frac{\partial^3 u_1}{\partial x_1 \partial x_2^2} + \frac{\partial^3 u_2}{\partial x_2 \partial x_1^2} = \frac{\partial^2}{\partial x_1 \partial x_2} \left( \frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1} \right) = 2 \frac{\partial^2 \varepsilon_{12}}{\partial x_1 \partial x_2}$$

Similarly:

$$\frac{\partial^2 \varepsilon_{11}}{\partial x_3^2} + \frac{\partial^2 \varepsilon_{33}}{\partial x_1^2} = 2 \frac{\partial^2 \varepsilon_{13}}{\partial x_1 \partial x_3}$$

$$\frac{\partial^2 \varepsilon_{22}}{\partial x_3^2} + \frac{\partial^2 \varepsilon_{33}}{\partial x_2^2} = 2 \frac{\partial^2 \varepsilon_{23}}{\partial x_2 \partial x_3}$$

### Chap. 5 Mechanical behavior of solids

In a general boundary value problem in solid mechanics, the number of unknown variables is

$u_i$  (3),  $\varepsilon_{ij}$  (6),  $\sigma_{ij}$  (6)  $\rightarrow$  15 unknown variables

Number of equations:

Equilibrium equations:  $\sigma_{ij,j} + \rho_i = \ddot{u}_i$  (3 equations)

Strain-displacement equations:  $\varepsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i})$  (6 equations)

Therefore, we are still missing 6 equations to determine the 15 unknowns. The six additional equations to close the formulations can be generally expressed as:

$$\underline{\sigma} = f(\underline{\varepsilon})$$

This leads to the constitutive model of different material behaviors which can be categorized into the following types:

Linear elastic material (Hooke's law)

Elastic-plastic material

Visco-elastic material

Visco-plastic material

...

Linear elastic material:

We have seen Hooke's law in 1D:



$$\sigma = E\varepsilon$$

In 3D, one might generalize this in tensor form as

$$\underline{\sigma} = \underline{C}\underline{\varepsilon} \quad \text{or} \quad \sigma_{ij} = C_{ijkl}\varepsilon_{kl}$$

Where  $C_{ijkl}$  is a 4<sup>th</sup> order tensor. For the most general case, a 4<sup>th</sup> order tensor has  $3^4 = 81$

independent components. However,  $\underline{\sigma}$  and  $\underline{\varepsilon}$  are both symmetric tensors that only have six

independent components. Therefore,  $\underline{\sigma} = \underline{C}\underline{\varepsilon}$  can be rewritten in a matrix form as

$$\begin{pmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{13} \\ \sigma_{12} \end{pmatrix} = \begin{pmatrix} C_{11} & C_{12} & C_{13} & C_{14} & C_{15} & C_{16} \\ C_{21} & C_{22} & C_{23} & C_{24} & C_{25} & C_{26} \\ C_{31} & C_{32} & C_{33} & C_{34} & C_{35} & C_{36} \\ C_{41} & C_{42} & C_{43} & C_{44} & C_{45} & C_{46} \\ C_{51} & C_{52} & C_{53} & C_{54} & C_{55} & C_{56} \\ C_{61} & C_{62} & C_{63} & C_{64} & C_{65} & C_{66} \end{pmatrix} \begin{pmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ \varepsilon_{23} \\ \varepsilon_{13} \\ \varepsilon_{12} \end{pmatrix}$$

Clearly, the number of independent components of  $\underline{C}$  is reduced to 36.

Furthermore, the  $6 \times 6$  matrix of  $\underline{C}$  is symmetric, which further reduces the number of independent components to 21 (the reason will be discussed in the next lecture).