

<http://www.Drshokuhi.com>

سایت آموزش مهندسی مکانیک ایران

We continue on the mathematical background.

Base tensors: $\bar{e}_i \otimes \bar{e}_j$ (dyadic form)

$$\bar{e}_1 \otimes \bar{e}_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \bar{e}_1 \otimes \bar{e}_2 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \bar{e}_1 \otimes \bar{e}_3 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \dots$$

Note that: $\bar{e}_i \otimes \bar{e}_j \neq \bar{e}_j \otimes \bar{e}_i$, $\bar{e}_i \otimes \bar{e}_j \neq (\bar{e}_j \otimes \bar{e}_i)^T$

Rules of operation:

$$(\bar{e}_i \otimes \bar{e}_j) \bar{e}_k = \bar{e}_i (\bar{e}_j \cdot \bar{e}_k) = \delta_{jk} \bar{e}_i$$

$$\bar{e}_k \cdot (\bar{e}_i \otimes \bar{e}_j) = (\bar{e}_k \cdot \bar{e}_i) \bar{e}_j = \delta_{ki} \bar{e}_j$$

Tensor product of vectors (also called tensor dyad):

$$\bar{a} \otimes \bar{b} = (a_i \bar{e}_i) \otimes (b_j \bar{e}_j) = a_i b_j (\bar{e}_i \otimes \bar{e}_j) = \begin{bmatrix} a_1 b_1 & a_1 b_2 & a_1 b_3 \\ a_2 b_1 & a_2 b_2 & a_2 b_3 \\ a_3 b_1 & a_3 b_2 & a_3 b_3 \end{bmatrix}$$

We can compare the matrix forms of different vector products:

$$\bar{a} \cdot \bar{b} = \begin{bmatrix} a_1 & a_2 & a_3 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = a_1 b_1 + a_2 b_2 + a_3 b_3$$

$$\bar{a} \otimes \bar{b} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \begin{bmatrix} b_1 & b_2 & b_3 \end{bmatrix} = \begin{bmatrix} a_1 b_1 & a_1 b_2 & a_1 b_3 \\ a_2 b_1 & a_2 b_2 & a_2 b_3 \\ a_3 b_1 & a_3 b_2 & a_3 b_3 \end{bmatrix}$$

In general, $\underline{T} = T_{ij} \bar{e}_i \otimes \bar{e}_j$

$$\begin{bmatrix} T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{23} \\ T_{31} & T_{32} & T_{33} \end{bmatrix} = T_{11} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + T_{12} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + T_{13} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \dots$$

$$T_{ij} = \bar{e}_i \cdot \underline{T} \bar{e}_j$$

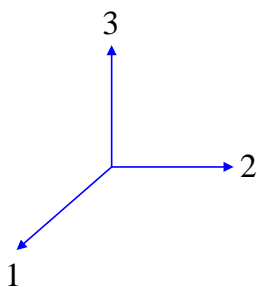
For example,

$$T_{11} = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{23} \\ T_{31} & T_{32} & T_{33} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

One can also verify this using index notation:

$$T_{ij} = \bar{e}_i \cdot (T_{pq} \bar{e}_p \otimes \bar{e}_q) \bar{e}_j = T_{pq} (\bar{e}_i \cdot \bar{e}_p) (\bar{e}_q \cdot \bar{e}_j) = T_{pq} \delta_{ip} \delta_{qj} = T_{ij}$$

This definition is very useful in practice in deriving the components of a tensor from its physical descriptions.



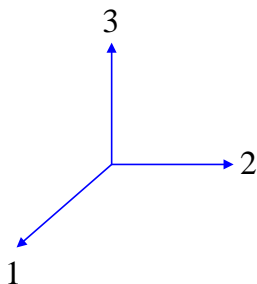
Example 1: Find the tensor \underline{R} that reflects any vectors with respect to the 2-3 plane.

Solution: According to the definition of reflection, we have

$$\underline{R} \bar{e}_1 = -\bar{e}_1, \quad \underline{R} \bar{e}_2 = \bar{e}_2, \quad \underline{R} \bar{e}_3 = \bar{e}_3$$

Therefore:

$$R_{ij} = \bar{e}_i \cdot \underline{R} \bar{e}_j = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$



Example 2: Find the tensor \underline{Q} that rotates any vector about the \underline{x}_3 axis by 90° clockwise.

$$\underline{Q} \bar{e}_1 = -\bar{e}_2, \quad \underline{Q} \bar{e}_2 = \bar{e}_1, \quad \underline{Q} \bar{e}_3 = \bar{e}_3$$

Therefore:

$$Q_{ij} = \bar{e}_i \cdot \underline{Q} \bar{e}_j = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Symmetric tensors:

$$\underline{T} = \begin{bmatrix} T_{11} & T_{12} & T_{13} \\ T_{12} & T_{22} & T_{23} \\ T_{13} & T_{23} & T_{33} \end{bmatrix} = \underline{T}^T$$

Antisymmetric tensors:

$$\underline{A} = \begin{bmatrix} 0 & A_{12} & A_{13} \\ -A_{12} & 0 & A_{23} \\ -A_{13} & -A_{23} & 0 \end{bmatrix} = -\underline{A}^T$$

An arbitrary tensor \underline{P} can be formed into a sum of a symmetric tensor and an antisymmetric tensor:

$$\underline{P} = \frac{\underline{P} + \underline{P}^T}{2} + \frac{\underline{P} - \underline{P}^T}{2} = \underline{P}^S + \underline{P}^A$$

Eigenvalues of a tensor:

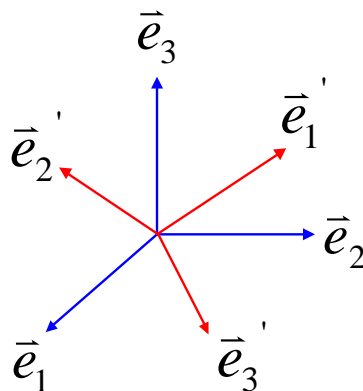
$$\underline{T} \underline{u} = \lambda \underline{u}$$

where \underline{u} is called the eigenvectors of \underline{T} . λ is the eigenvalue.

$$\det(\underline{T} - \lambda \underline{I}) = 0$$

For a second rank tensor in 3 dimension, there are 3 eigenvalues and 3 eigenvectors.

Tensor transformation under change of coordinates:



In different coordinates,

$$\underline{T} = T_{ij} \bar{e}_i \otimes \bar{e}_j = \begin{bmatrix} T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{23} \\ T_{31} & T_{32} & T_{33} \end{bmatrix}$$

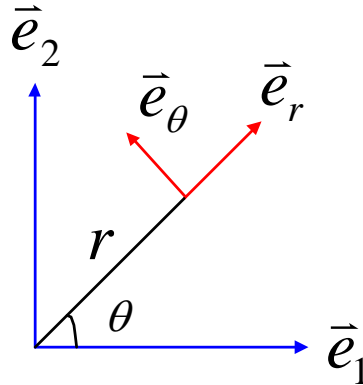
$$\underline{T} = T'_{pq} \bar{e}'_p \otimes \bar{e}'_q = \begin{bmatrix} T'_{11} & T'_{12} & T'_{13} \\ T'_{21} & T'_{22} & T'_{23} \\ T'_{31} & T'_{32} & T'_{33} \end{bmatrix}$$

How are T_{ij} and T'_{ij} related?

$$\underline{T} = T_{ij} \bar{e}_i \otimes \bar{e}_j = T'_{pq} \bar{e}'_p \otimes \bar{e}'_q$$

$$T'_{pq} = \bar{e}'_p \cdot \underline{T} \bar{e}'_q = \bar{e}'_p \cdot (T_{ij} \bar{e}_i \otimes \bar{e}_j) \bar{e}'_q = T_{ij} (\bar{e}'_p \cdot \bar{e}_i) (\bar{e}'_q \cdot \bar{e}_j)$$

Example: 2D Cartesian coordinate and Polar coordinate.



$$\begin{aligned} \underline{T} &= T_{11} \bar{e}_1 \otimes \bar{e}_1 + T_{12} \bar{e}_1 \otimes \bar{e}_2 + T_{21} \bar{e}_2 \otimes \bar{e}_1 + T_{22} \bar{e}_2 \otimes \bar{e}_2 \\ &= T_{rr} \bar{e}_r \otimes \bar{e}_r + T_{r\theta} \bar{e}_r \otimes \bar{e}_\theta + T_{\theta r} \bar{e}_\theta \otimes \bar{e}_r + T_{\theta\theta} \bar{e}_\theta \otimes \bar{e}_\theta \end{aligned}$$

$$\begin{aligned} T_{11} &= \bar{e}_1 \cdot \underline{T} \bar{e}_1 = \bar{e}_1 \cdot (T_{rr} \bar{e}_r \otimes \bar{e}_r + T_{r\theta} \bar{e}_r \otimes \bar{e}_\theta + T_{\theta r} \bar{e}_\theta \otimes \bar{e}_r + T_{\theta\theta} \bar{e}_\theta \otimes \bar{e}_\theta) \bar{e}_1 \\ &= T_{rr} \cos^2 \theta - T_{r\theta} \cos \theta \sin \theta - T_{\theta r} \cos \theta \sin \theta + T_{\theta\theta} \sin^2 \theta \end{aligned}$$

$$T_{12} = T_{rr} \cos \theta \sin \theta + T_{r\theta} \cos^2 \theta - T_{\theta r} \sin^2 \theta - T_{\theta\theta} \cos \theta \sin \theta$$

Tensor equations:

Example: Generalized Hooke's Law

$$\varepsilon_{ij} = \frac{1+\nu}{E} \sigma_{ij} - \frac{\nu}{E} \sigma_{kk} \delta_{ij}$$

Inverting this equation by letting $j \rightarrow i$,

$$\varepsilon_{ii} = \frac{1+\nu}{E} \sigma_{ii} - \frac{\nu}{E} \sigma_{kk} \delta_{ii} = \frac{1-2\nu}{E} \sigma_{ii}$$

i.e.

$$\sigma_{kk} = \frac{E}{1-2\nu} \varepsilon_{kk}$$

Substituting σ_{kk} into the original equation,

$$\varepsilon_{ij} = \frac{1+\nu}{E} \sigma_{ij} - \frac{\nu}{E} \frac{E}{1-2\nu} \varepsilon_{kk} \delta_{ij}$$

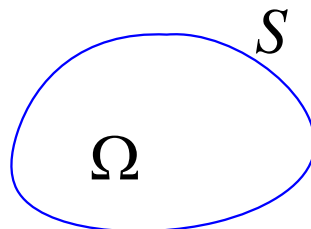
$$\frac{1+\nu}{E} \sigma_{ij} = \varepsilon_{ij} + \frac{\nu}{1-2\nu} \varepsilon_{kk} \delta_{ij}$$

$$\begin{aligned} \sigma_{ij} &= \frac{E}{1+\nu} \varepsilon_{ij} + \frac{\nu E}{(1+\nu)(1-2\nu)} \varepsilon_{kk} \delta_{ij} \\ &= 2\mu \varepsilon_{ij} + \lambda \varepsilon_{kk} \delta_{ij} \end{aligned}$$

E, ν : Young's modulus and Poisson's ratio

$\mu = \frac{E}{2(1+\nu)}$, $\lambda = \frac{\nu E}{(1+\nu)(1-2\nu)}$ are called Lamé constants.

Divergence theorem:



$$\int_{\Omega} (\nabla \cdot \vec{v}) dv = \oint_S \vec{v} \cdot \vec{n} dS$$

Generalization:

$$\int_{\Omega} (\cdot)_{,i} dv = \oint_S (\cdot) n_i dS$$

(\cdot) can be any vector or tensor field.