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سایت آموزش مهندسی مکانیک ایران

Continuing on concepts of stress in a continuum:

Traction on an arbitrary plane with normal vector  $\vec{n}$

$$t_j = \sigma_{ij} n_i$$

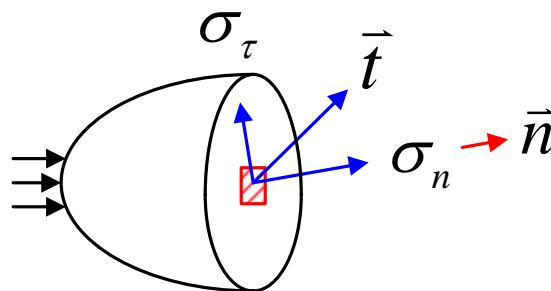
$$\vec{t} = \underline{\sigma}^T \vec{n}$$

This equation indicates that the normal stress on a plane with normal vector  $\vec{n}$  is

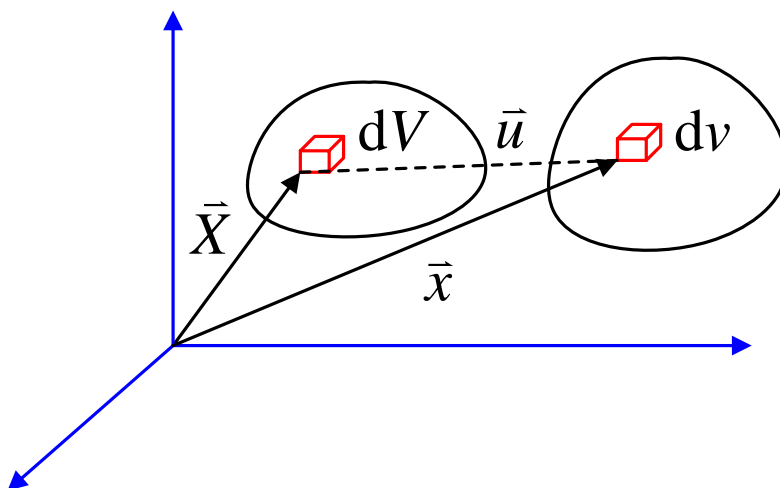
$$\sigma_n = \vec{t} \cdot \vec{n} = \vec{n} \cdot \underline{\sigma}^T \vec{n}$$

Accordingly, the shear stress on a plane with normal  $\vec{n}$  is

$$\sigma_\tau \vec{\hat{t}} = \vec{t} - \sigma_n \vec{n} = \underline{\sigma}^T \vec{n} - (\vec{n} \cdot \underline{\sigma}^T \vec{n}) \vec{n}$$



**Force equilibrium** (i.e. application of Newton's law  $\vec{F} = m\vec{a}$  to a continuum; also called balance of linear momentum)



$\bar{u}$  — displacement vector

$\frac{\partial \bar{u}}{\partial t} = \dot{\bar{u}}$  — velocity vector

$\frac{\partial^2 \bar{u}}{\partial t^2} = \ddot{\bar{u}}$  — acceleration vector

$\int_V \rho \ddot{\bar{u}} dV$  — total inertia force on volume  $V$  ( $dm = \rho dV$ ,  $a = \ddot{\bar{u}}$ )

Consider all the forces acting on the volume:

$\int_S \bar{t} dS$  — total surface force on  $V$

$\int_V \bar{f} dV$  — total body force on  $V$

According to Newton's law,

$$\int_S \bar{t} dS + \int_V \bar{f} dV = \int_V \rho \ddot{\bar{u}} dV$$

In index notation:

$$\int_S t_j dS + \int_V f_j dV = \int_V \rho \ddot{u}_j dV$$

The first term is

$$\int_S t_j dS = \int_S \sigma_{ij} n_i dS = \int_V \sigma_{ij,i} dV \quad (\text{Divergence theorem})$$

Thus

$$\int_V (\sigma_{ij,i} + f_j - \rho \ddot{u}_j) dV = 0$$

Since this is true for any volume  $V$  (e.g., we can shrink  $V$  to a point), the integrand must vanish every point in the continuum,

$$\sigma_{ij,i} + f_j = \rho \ddot{u}_j$$

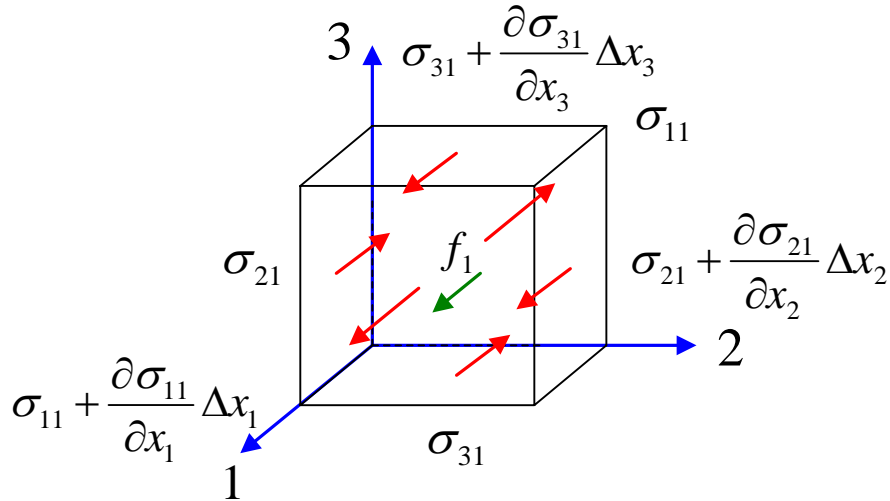
i.e.

$$\frac{\partial \sigma_{ij}}{\partial x_i} + f_j = \rho \ddot{u}_j$$

Compare this with the equation we derived earlier for the 1D case:  $\frac{\partial \sigma}{\partial x} + f = \rho \ddot{u}$

Classical derivation of equilibrium equations for a continuum:

Consider all the forces in the  $x_1$  acting on a small block of material shown below:



Summing up all the forces in  $x_1$  direction and applying Newton's law,

$$\left( \sigma_{11} + \frac{\partial \sigma_{11}}{\partial x_1} \Delta x_1 - \sigma_{11} \right) \Delta x_2 \Delta x_3 + \left( \sigma_{21} + \frac{\partial \sigma_{21}}{\partial x_2} \Delta x_2 - \sigma_{21} \right) \Delta x_1 \Delta x_3 + \left( \sigma_{31} + \frac{\partial \sigma_{31}}{\partial x_3} \Delta x_3 - \sigma_{31} \right) \Delta x_1 \Delta x_2 + f_1 \Delta x_1 \Delta x_2 \Delta x_3 = \rho \Delta x_1 \Delta x_2 \Delta x_3 \ddot{u}_1$$

This leads to

$$\frac{\partial \sigma_{11}}{\partial x_1} + \frac{\partial \sigma_{21}}{\partial x_2} + \frac{\partial \sigma_{31}}{\partial x_3} + f_1 = \rho \ddot{u}_1$$

Similarly,

$$\frac{\partial \sigma_{12}}{\partial x_1} + \frac{\partial \sigma_{22}}{\partial x_2} + \frac{\partial \sigma_{32}}{\partial x_3} + f_2 = \rho \ddot{u}_2$$

$$\frac{\partial \sigma_{13}}{\partial x_1} + \frac{\partial \sigma_{23}}{\partial x_2} + \frac{\partial \sigma_{33}}{\partial x_3} + f_3 = \rho \ddot{u}_3$$

i.e.

$$\frac{\partial \sigma_{ij}}{\partial x_i} + f_j = \rho \ddot{u}_j$$

### Moment equilibrium (Balance of angular momentum)

Net moment of all forces should vanish at equilibrium

$$\int_S \bar{x} \times \bar{t} \, dS + \int_V \bar{x} \times \bar{f} \, dV = \int_V \bar{x} \times \rho \ddot{\bar{u}} \, dV$$

In index notation:

$$\int_S \varepsilon_{ijk} x_i t_j \bar{e}_k \, dS + \int_V \varepsilon_{ijk} x_i f_j \bar{e}_k \, dV = \int_V \varepsilon_{ijk} x_i \rho \ddot{u}_j \bar{e}_k \, dV$$

Canceling  $\bar{e}_k$  at both sides,

$$\int_S \varepsilon_{ijk} x_i t_j \, dS + \int_V \varepsilon_{ijk} x_i f_j \, dV = \int_V \varepsilon_{ijk} x_i \rho \ddot{u}_j \, dV$$

The first term is

$$\begin{aligned} \int_S \varepsilon_{ijk} x_i \sigma_{pj} n_p \, dS &= \int_V (\varepsilon_{ijk} x_i \sigma_{pj})_{,p} \, dV \\ &= \int_V \varepsilon_{ijk} (\delta_{ip} \sigma_{pj} + x_i \sigma_{pj,p}) \, dV \\ &= \int_V \varepsilon_{ijk} \sigma_{ij} + \varepsilon_{ijk} x_i \sigma_{pj,p} \, dV \end{aligned}$$

Thus

$$\begin{aligned} \int_V [\varepsilon_{ijk} \sigma_{ij} + \varepsilon_{ijk} x_i (\sigma_{pj,p} + f_j - \rho \ddot{u}_j)] \, dV &= 0 \\ \Rightarrow \varepsilon_{ijk} \sigma_{ij} &= 0 \end{aligned}$$

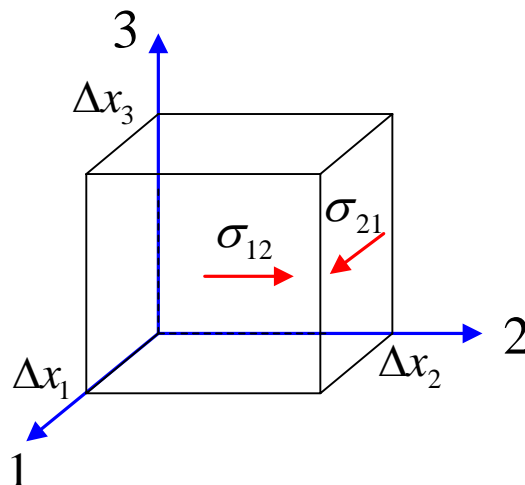
$$k = 1, \quad \varepsilon_{ij1} \sigma_{ij} = \sigma_{23} - \sigma_{32} = 0 \Rightarrow \sigma_{23} = \sigma_{32}$$

$$k = 2, \quad \varepsilon_{ij2} \sigma_{ij} = \sigma_{31} - \sigma_{13} = 0 \Rightarrow \sigma_{31} = \sigma_{13}$$

$$k = 3, \quad \varepsilon_{ij3} \sigma_{ij} = \sigma_{12} - \sigma_{21} = 0 \Rightarrow \sigma_{12} = \sigma_{21}$$

Therefore, balance of angular momentum states that stress tensor is symmetric, i.e.

$$\sigma_{ij} = \sigma_{ji}$$



A simpler demonstration of symmetry of stress tensor:

$$(\sigma_{12}\Delta x_2\Delta x_3) \cdot \Delta x_1 = (\sigma_{21}\Delta x_1\Delta x_3) \cdot \Delta x_2$$

$$\Rightarrow \sigma_{12} = \sigma_{21}$$

$$t_i = \sigma_{ji}n_j$$

$$\vec{t} = \underline{\sigma}^T \vec{n} = \underline{\sigma} \vec{n} \quad (\underline{\sigma} \text{ is symmetric})$$

### Eigenvalues of a matrix

$$\underline{T} \vec{\xi} = \lambda \vec{\xi}$$

This has a very clear meaning for stress tensors. The eigenvalues of a stress tensor are called principal stresses, corresponding to the normal stresses on planes with no shear stresses, i.e.

$$\vec{t} = \underline{\sigma} \vec{n} = \sigma \vec{n}$$

There in general exist 3 principal stresses and 3 mutually orthogonal principal directions.

Writing  $\underline{\sigma} \vec{n} = \sigma \vec{n}$  in matrix form:

$$\begin{bmatrix} \sigma_{11} - \sigma & \sigma_{12} & \sigma_{13} \\ \sigma_{12} & \sigma_{22} - \sigma & \sigma_{23} \\ \sigma_{13} & \sigma_{23} & \sigma_{33} - \sigma \end{bmatrix} \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix} = 0$$

For nontrivial solution of  $\vec{n}$ ,

$$\begin{bmatrix} \sigma_{11} - \sigma & \sigma_{12} & \sigma_{13} \\ \sigma_{12} & \sigma_{22} - \sigma & \sigma_{23} \\ \sigma_{13} & \sigma_{23} & \sigma_{33} - \sigma \end{bmatrix} = 0 \quad (*)$$

Let the solution to equation (\*) be  $\sigma_1 \neq \sigma_2 \neq \sigma_3$ ,

$$\underline{\sigma} \vec{n}^{(1)} = \sigma_1 \vec{n}^{(1)} \quad (1)$$

$$\underline{\sigma} \bar{n}^{(2)} = \sigma_2 \bar{n}^{(2)} \quad (2)$$

$\bar{n}^{(2)} \cdot (1) - \bar{n}^{(1)} \cdot (2)$  leads to

$$(\sigma_1 - \sigma_2) \bar{n}^{(1)} \cdot \bar{n}^{(2)} = 0$$

which shows that, if  $\sigma_1 \neq \sigma_2$ , we must have  $\bar{n}^{(1)} \cdot \bar{n}^{(2)} = 0$ . Therefore,  $\bar{n}^{(1)}$ ,  $\bar{n}^{(2)}$  are orthogonal vectors,  $\bar{n}^{(1)} \perp \bar{n}^{(2)}$ . Similarly, we can show  $\bar{n}^{(1)} \perp \bar{n}^{(3)}$  and  $\bar{n}^{(2)} \perp \bar{n}^{(3)}$ .