

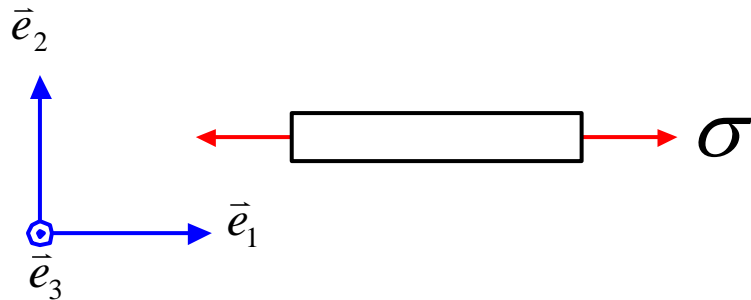
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سایت آموزش مهندسی مکانیک ایران

## Stress in a solid

### Examples of common stress states:

#### 1. Uniaxial stress

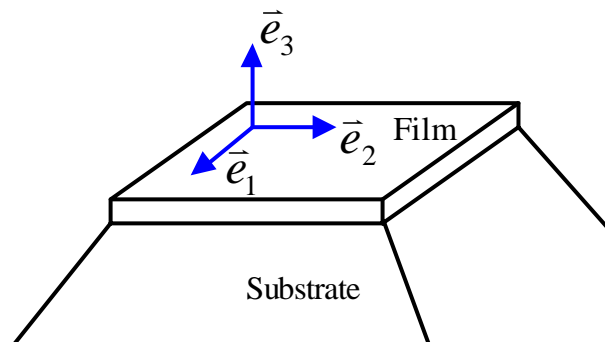


$$\underline{\sigma} = \begin{bmatrix} \sigma & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Principal stresses:  $\sigma_I = \sigma$ ,  $\sigma_{II} = \sigma_{III} = 0$

Principal directions:  $\bar{e}_1, \bar{e}_2, \bar{e}_3$  ( $\bar{e}_2, \bar{e}_3$  can be replaced by any direction in 2-3 plane)

#### 2. Biaxial stress

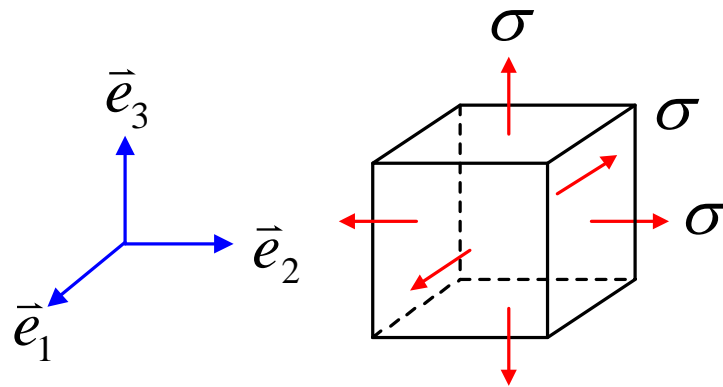


$$\underline{\sigma} = \begin{bmatrix} \sigma & 0 & 0 \\ 0 & \sigma & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Principal stresses:  $\sigma_I = \sigma_{II} = \sigma$ ,  $\sigma_{III} = 0$

Principal directions:  $\bar{e}_1, \bar{e}_2, \bar{e}_3$  ( $\bar{e}_1, \bar{e}_2$  can be replaced by any direction in 1-2 plane)

## 3. Hydrostatic stress

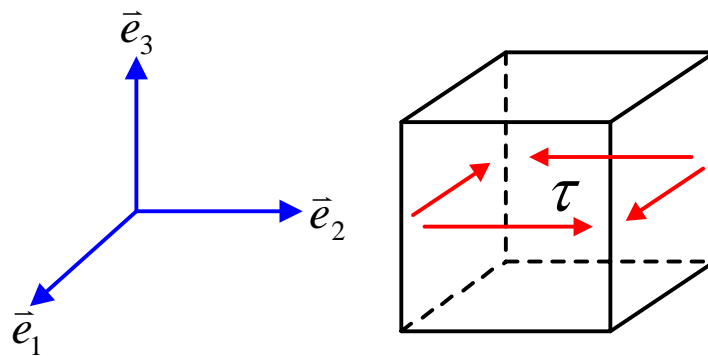


$$\underline{\sigma} = \begin{bmatrix} \sigma & 0 & 0 \\ 0 & \sigma & 0 \\ 0 & 0 & \sigma \end{bmatrix} = \sigma \underline{I}$$

Principal stresses:  $\sigma_I = \sigma_{II} = \sigma_{III} = \sigma$

Principal directions: any direction (since there is no shear stress on any cut plane)

## 4. Pure shear



$$\underline{\sigma} = \begin{bmatrix} 0 & \tau & 0 \\ \tau & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \tau \bar{e}_1 \otimes \bar{e}_2 + \tau \bar{e}_2 \otimes \bar{e}_1$$

Principal stresses:

$$\begin{bmatrix} 0 & \tau & 0 \\ \tau & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix} = \lambda \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix} \Rightarrow \begin{bmatrix} -\lambda & \tau & 0 \\ \tau & -\lambda & 0 \\ 0 & 0 & -\lambda \end{bmatrix} \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix} = 0$$

For nontrivial solution of  $n_i$ ,

$$\det \begin{bmatrix} -\lambda & \tau & 0 \\ \tau & -\lambda & 0 \\ 0 & 0 & -\lambda \end{bmatrix} = 0,$$

which indicates

$$-\lambda(\lambda^2 - \tau^2) = 0 \Rightarrow \lambda = \tau, 0, -\tau$$

Following the convention  $\sigma_I > \sigma_{II} > \sigma_{III}$ , we can write

$$\sigma_I = \tau, \quad \sigma_{II} = 0, \quad \sigma_{III} = -\tau$$

Principal directions:

Principal direction  $I$  ( $\sigma_I = \tau$ ):

$$\begin{bmatrix} -\tau & \tau & 0 \\ \tau & -\tau & 0 \\ 0 & 0 & -\tau \end{bmatrix} \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix} = 0$$

That is

$$n_1 = n_2, \quad n_3 = 0$$

In addition,  $n_1^2 + n_2^2 + n_3^2 = 1$  (because  $\bar{n}$  is a unit vector).

$$\begin{cases} n_1 = \pm \frac{1}{\sqrt{2}} \\ n_2 = \pm \frac{1}{\sqrt{2}} \\ n_3 = 0 \end{cases}$$

Therefore:  $\bar{n}_I = \pm \left( \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0 \right)$

Principal direction  $II$  ( $\sigma_{II} = 0$ ):

$$\begin{bmatrix} 0 & \tau & 0 \\ \tau & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix} = 0$$

That is

$$n_1 = n_2 = 0$$

In addition,  $n_1^2 + n_2^2 + n_3^2 = 1$  ( $\bar{n}$  is a unit vector).

$$\begin{cases} n_1 = 0 \\ n_2 = 0 \\ n_3 = \pm 1 \end{cases}$$

Therefore:  $\bar{n}_{II} = \pm(0, 0, 1)$

Principal direction *III* ( $\sigma_{III} = -\tau$ ):

$$\begin{bmatrix} \tau & \tau & 0 \\ \tau & \tau & 0 \\ 0 & 0 & \tau \end{bmatrix} \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix} = 0$$

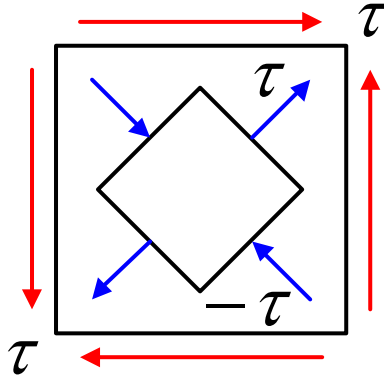
That is

$$n_1 = -n_2, \quad n_3 = 0$$

In addition,  $n_1^2 + n_2^2 + n_3^2 = 1$  ( $\bar{n}$  is a unit vector).

$$\begin{cases} n_1 = \pm \frac{1}{\sqrt{2}} \\ n_2 = \mp \frac{1}{\sqrt{2}} \\ n_3 = 0 \end{cases}$$

Therefore:  $\bar{n}_{III} = \pm\left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 0\right)$



### Deviatoric stress tensor

An arbitrary stress tensor can be written in

$$\sigma_{ij} = \frac{1}{3} \sigma_{kk} \delta_{ij} + \sigma'_{ij}$$

Define  $\sigma_h = \frac{1}{3} \sigma_{kk} = \frac{1}{3} (\sigma_I + \sigma_{II} + \sigma_{III})$ , which represents the hydrostatic part of the stress.

$$\sigma'_{ij} = \sigma_{ij} - \sigma_h \delta_{ij}$$

Let  $i \rightarrow j$ ,

$$\sigma'_{jj} = \sigma_{jj} - \sigma_h \delta_{jj} = \sigma_{jj} - 3\sigma_h = 0$$

i.e.  $\text{trace}(\sigma'_{ij}) = \sigma'_{11} + \sigma'_{22} + \sigma'_{33} = 0$ .

$\sigma'_{ij}$  has 3 invariants:  $\sigma'_{kk} = 0$ ,  $\sigma'_{ij} \sigma'_{ij}$ ,  $\det(\sigma'_{ij})$ .

Among the 3 invariants of  $\sigma'_{ij}$ ,  $\sigma'_{ij} \sigma'_{ij}$  is related to the von Mises stress which plays an important role in modeling plastic yielding).

### von Mises stress

Consider the case of uniaxial tension with principal stresses:

$$\sigma_I = \sigma, \quad \sigma_{II} = \sigma_{III} = 0$$

The stress tensor in the principal directions is

$$\underline{\sigma} = \begin{bmatrix} \sigma & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

The principal stresses of the deviatoric stress tensor are

$$\sigma'_I = \frac{2}{3} \sigma, \quad \sigma'_{II} = \sigma'_{III} = -\frac{1}{3} \sigma$$

Therefore

$$\sigma'_{ij}\sigma'_{ij} = \sigma_I'^2 + \sigma_{II}'^2 + \sigma_{III}'^2 = \frac{2}{3}\sigma^2$$

i.e.

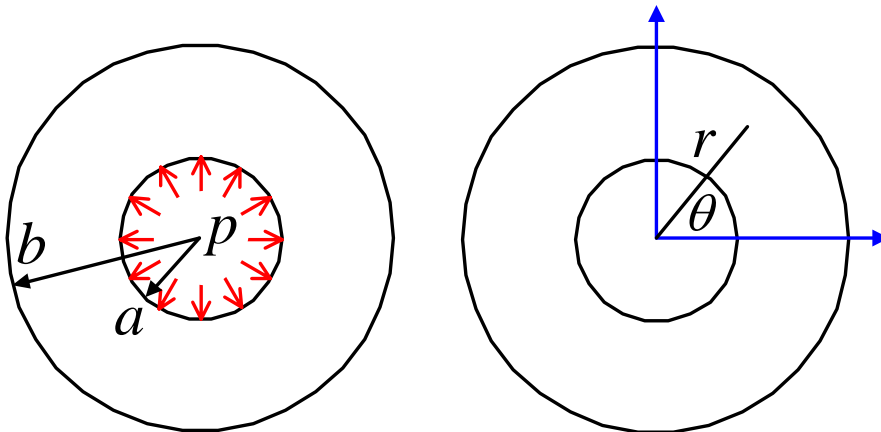
$$\sqrt{\frac{3}{2}\sigma'_{ij}\sigma'_{ij}} = \sigma$$

von Mises stress is defined as  $\sigma_e = \sqrt{\frac{3}{2}\sigma'_{ij}\sigma'_{ij}}$  such that  $\sigma_e = \sigma$  in the case of uniaxial tension.

### Traction BC's for BVPs in solid mechanics

The Cauchy relationship between stress and traction serves as a basis to express traction boundary conditions in terms of the relevant stress components. We consider two examples

Example 1:



$$\bar{t} = \underline{\sigma} \bar{e}_r = 0 \quad @ \quad r = b \quad (\text{traction free at } r = b)$$

$$\bar{t} = \underline{\sigma}(-\bar{e}_r) = p \bar{e}_r \quad @ \quad r = a$$

In Polar coordinates,

@  $r = b$ :

$$\sigma_{rr} = \bar{e}_r \cdot \underline{\sigma} \bar{e}_r = 0, \quad \sigma_{r\theta} = \bar{e}_\theta \cdot \underline{\sigma} \bar{e}_r = 0$$

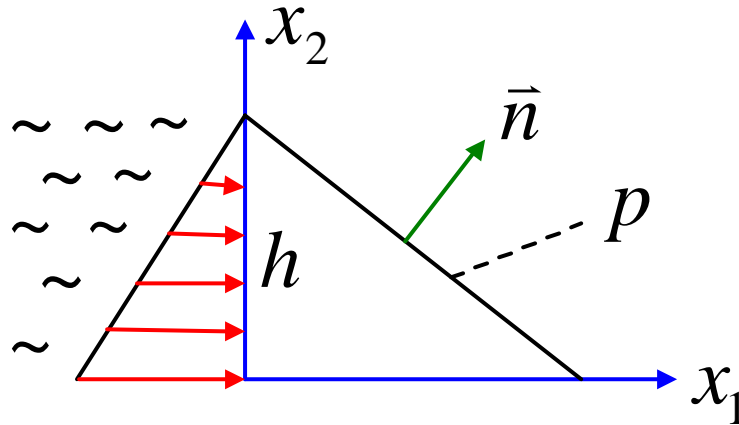
@  $r = a$ :

$$\sigma_{rr} = \bar{e}_r \cdot \underline{\sigma} \bar{e}_r = -p, \quad \sigma_{r\theta} = \bar{e}_\theta \cdot \underline{\sigma} \bar{e}_r = 0$$

Therefore, the BCs for the above problem are often expressed as

$$\begin{cases} \sigma_{rr} = -p & \sigma_{r\theta} = 0 & r = a \\ \sigma_{rr} = 0 & \sigma_{r\theta} = 0 & r = a \end{cases}$$

Example 2:



$$\bar{t} = \underline{\sigma}(-\bar{e}_1) = \rho g(h-x_2)\bar{e}_1 \quad @ \quad x_1 = 0$$

$$\bar{t} = \underline{\sigma}\bar{n} = 0 \quad @ \quad p \text{ plane}$$

$$@ \quad x_1 = 0$$

$$\sigma_{11} = \bar{e}_1 \cdot \underline{\sigma} \bar{e}_1 = -\rho g(h-x_2)$$

$$\sigma_{12} = \bar{e}_2 \cdot \underline{\sigma} \bar{e}_1 = 0$$

### Maximum and minimum normal & shear stresses in a solid

We want to show that, given principal stresses  $\sigma_I > \sigma_{II} > \sigma_{III}$ , the maximum and minimum normal and shear stresses in a solid have the following simple results:

$$\begin{aligned} \max(\sigma_n) &= \sigma_I, \quad \min(\sigma_n) = \sigma_{III} \\ \max(\sigma_s) &= \frac{\sigma_I - \sigma_{III}}{2}, \quad \min(\sigma_s) = 0 \end{aligned}$$

Proof:

Choose base vectors in the principal directions,

$$\underline{\sigma} = \begin{bmatrix} \sigma_I & 0 & 0 \\ 0 & \sigma_{II} & 0 \\ 0 & 0 & \sigma_{III} \end{bmatrix}$$

The traction on an arbitrary plane with normal  $\bar{n}$  (note that  $n_1^2 + n_2^2 + n_3^2 = 1$ ) is

$$\bar{t} = \underline{\sigma} \bar{n}$$

The normal stress on that plane is

$$\sigma_n = \bar{n} \cdot \bar{t} = \bar{n} \cdot \underline{\sigma} \bar{n} = \sigma_I n_1^2 + \sigma_{II} n_2^2 + \sigma_{III} n_3^2$$

Introduce Lagrangian multiplier  $\lambda$ ,

$$F = \sigma_I n_1^2 + \sigma_{II} n_2^2 + \sigma_{III} n_3^2 - \lambda (n_1^2 + n_2^2 + n_3^2)$$

$$\frac{\partial F}{\partial n_1} = 0 \Rightarrow 2\sigma_I n_1 - 2\lambda n_1 = 0 \quad (1)$$

$$\frac{\partial F}{\partial n_2} = 0 \Rightarrow 2\sigma_{II} n_2 - 2\lambda n_2 = 0 \quad (2)$$

$$\frac{\partial F}{\partial n_3} = 0 \Rightarrow 2\sigma_{III} n_3 - 2\lambda n_3 = 0 \quad (3)$$

Case 1:  $n_1 \neq 0$ ,  $n_2 \neq 0$ ,  $n_3 \neq 0$

From the equations (1)-(3):

$$\sigma_I = \sigma_{II} = \sigma_{III} = \lambda,$$

which is contradictory to our assumption  $\sigma_I > \sigma_{II} > \sigma_{III}$ . (The cases when some of the principal stresses are equal are quite simple and can be treated separately.)

Case 2: 2 of  $n_1$ ,  $n_2$ ,  $n_3$  are not zero

If  $n_1 = 0$ ,  $n_2 \neq 0$ ,  $n_3 \neq 0$

$$2\sigma_{II} n_2 - 2\lambda n_2 = 0$$

$$2\sigma_{III} n_3 - 2\lambda n_3 = 0$$

$$n_2^2 + n_3^2 = 1$$

The solution is  $\sigma_{II} = \sigma_{III} = \lambda$ , again contradictory to our assumption  $\sigma_I > \sigma_{II} > \sigma_{III}$ .

Similarly,  $(n_2 = 0, n_1 \neq 0, n_3 \neq 0)$  and  $(n_3 = 0, n_1 \neq 0, n_2 \neq 0)$  also lead to solutions contradictory to our assumption.

Case 3: 2 of  $n_1, n_2, n_3$  are zero

If  $n_1 \neq 0, n_2 = n_3 = 0, \bar{n} = \pm(1, 0, 0)$

If  $n_2 \neq 0, n_1 = n_3 = 0, \bar{n} = \pm(0, 1, 0)$

If  $n_3 \neq 0, n_1 = n_2 = 0, \bar{n} = \pm(0, 0, 1)$

These are just principal directions. Therefore,

$$\max(\sigma_n) = \sigma_I, \quad \min(\sigma_n) = \sigma_{III} \quad (\sigma_I > \sigma_{II} > \sigma_{III})$$

