Chapter 2
HEAT CONDUCTION EQUATION

Introduction

2-1C Heat transfer is a *vector* quantity since it has direction as well as magnitude. Therefore, we must specify both direction and magnitude in order to describe heat transfer completely at a point. Temperature, on the other hand, is a scalar quantity.

2-2C The term *steady* implies *no change with time* at any point within the medium while *transient* implies *variation with time* or *time dependence.* Therefore, the temperature or heat flux remains unchanged with time during steady heat transfer through a medium at any location although both quantities may vary from one location to another. During transient heat transfer, the temperature and heat flux may vary with time as well as location. Heat transfer is one-dimensional if it occurs primarily in one direction. It is two-dimensional if heat tranfer in the third dimension is negligible.

2-3C Heat transfer to a canned drink can be modeled as two-dimensional since temperature differences (and thus heat transfer) will exist in the radial and axial directions (but there will be symmetry about the center line and no heat transfer in the azimuthal direction. This would be a transient heat transfer process since the temperature at any point within the drink will change with time during heating. Also, we would use the cylindrical coordinate system to solve this problem since a cylinder is best described in cylindrical coordinates. Also, we would place the origin somewhere on the center line, possibly at the center of the bottom surface.

2-4C Heat transfer to a potato in an oven can be modeled as one-dimensional since temperature differences (and thus heat transfer) will exist in the radial direction only because of symmetry about the center point. This would be a transient heat transfer process since the temperature at any point within the potato will change with time during cooking. Also, we would use the spherical coordinate system to solve this problem since the entire outer surface of a spherical body can be described by a constant value of the radius in spherical coordinates. We would place the origin at the center of the potato.

2-5C Assuming the egg to be round, heat transfer to an egg in boiling water can be modeled as one-dimensional since temperature differences (and thus heat transfer) will primarily exist in the radial direction only because of symmetry about the center point. This would be a transient heat transfer process since the temperature at any point within the egg will change with time during cooking. Also, we would use the spherical coordinate system to solve this problem since the entire outer surface of a spherical body can be described by a constant value of the radius in spherical coordinates. We would place the origin at the center of the egg.

2-6C Heat transfer to a hot dog can be modeled as two-dimensional since temperature differences (and thus heat transfer) will exist in the radial and axial directions (but there will be symmetry about the center line and no heat transfer in the azimuthal direction. This would be a transient heat transfer process since the temperature at any point within the hot dog will change with time during cooking. Also, we would use the cylindrical coordinate system to solve this problem since a cylinder is best described in cylindrical coordinates. Also, we would place the origin somewhere on the center line, possibly at the center of the hot dog. Heat transfer in a very long hot dog could be considered to be one-dimensional in preliminary calculations.

2-7C Heat transfer to a roast beef in an oven would be transient since the temperature at any point within the roast will change with time during cooking. Also, by approximating the roast as a spherical object, this heat transfer process can be modeled as one-dimensional since temperature differences (and thus heat transfer) will primarily exist in the radial direction because of symmetry about the center point.

2-8C Heat loss from a hot water tank in a house to the surrounding medium can be considered to be a steady heat transfer problem. Also, it can be considered to be two-dimensional since temperature differences (and thus heat transfer) will exist in the radial and axial directions (but there will be symmetry about the center line and no heat transfer in the azimuthal direction.)

2-9C Yes, the heat flux vector at a point P on an isothermal surface of a medium has to be perpendicular to the surface at that point.

2-10C Isotropic materials have the same properties in all directions, and we do not need to be concerned about the variation of properties with direction for such materials. The properties of anisotropic materials such as the fibrous or composite materials, however, may change with direction.

2-11C In heat conduction analysis, the conversion of electrical, chemical, or nuclear energy into heat (or thermal) energy in solids is called heat generation.

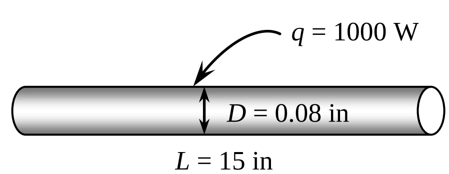
2-12C The phrase "thermal energy generation" is equivalent to "heat generation," and they are used interchangeably. They imply the conversion of some other form of energy into thermal energy. The phrase "energy generation," however, is vague since the form of energy generated is not clear.

2-13 Heat transfer through the walls, door, and the top and bottom sections of an oven is transient in nature since the thermal conditions in the kitchen and the oven, in general, change with time. However, we would analyze this problem as a steady heat transfer problem under the worst anticipated conditions such as the highest temperature setting for the oven, and the anticipated lowest temperature in the kitchen (the so called "design" conditions). If the heating element of the oven is large enough to keep the oven at the desired temperature setting under the presumed worst conditions, then it is large enough to do so under all conditions by cycling on and off.

Heat transfer from the oven is three-dimensional in nature since heat will be entering through all six sides of the oven. However, heat transfer through any wall or floor takes place in the direction normal to the surface, and thus it can be analyzed as being one-dimensional. Therefore, this problem can be simplified greatly by considering the heat transfer as being one- dimensional at each of the four sides as well as the top and bottom sections, and then by adding the calculated values of heat transfers at each surface.

2-14E The power consumed by the resistance wire of an iron is given. The heat generation and the heat flux are to be determined.

Assumptions Heat is generated uniformly in the resistance wire.



Analysis A 1000 W iron will convert electrical energy into heat in the wire at a rate of 1000 W. Therefore, the rate of heat generation in a resistance wire is simply equal to the power rating of a resistance heater. Then the rate of heat generation in the wire per unit volume is determined by dividing the total rate of heat generation by the volume of the wire to be

$$\dot{g} = \frac{\dot{G}}{V_{\text{wire}}} = \frac{\dot{G}}{(\pi D^2/4)L} = \frac{1000\,\text{W}}{[\pi(0.08/12\,\text{ft})^2/4](15/12\,\text{ft})}\left(\frac{3.412\,\text{Btu/h}}{1\,\text{W}}\right) = \mathbf{7.820 \times 10^7\ Btu/h \cdot ft^3}$$

Similarly, heat flux on the outer surface of the wire as a result of this heat generation is determined by dividing the total rate of heat generation by the surface area of the wire to be

$$\dot{q} = \frac{\dot{G}}{A_{\text{wire}}} = \frac{\dot{G}}{\pi DL} = \frac{1000\,\text{W}}{\pi(0.08/12\,\text{ft})(15/12\,\text{ft})}\left(\frac{3.412\,\text{Btu/h}}{1\,\text{W}}\right) = \mathbf{1.303 \times 10^5\ Btu/h \cdot ft^2}$$

Discussion Note that heat generation is expressed per unit volume in Btu/h·ft^3 whereas heat flux is expressed per unit surface area in Btu/h·ft^2.

2-15E

```
"GIVEN"
E_dot=1000 "[W]"
L=15 "[in]"
"D=0.08 [in], parameter to be varied"

"ANALYSIS"
g_dot=E_dot/V_wire*Convert(W, Btu/h)
V_wire=pi*D^2/4*L*Convert(in^3, ft^3)
q_dot=E_dot/A_wire*Convert(W, Btu/h)
A_wire=pi*D*L*Convert(in^2, ft^2)
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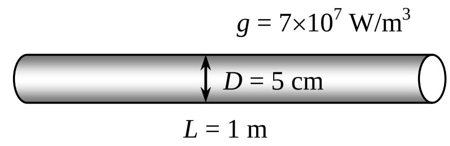
D [in]	q [Btu/h.ft^2]
0.02	521370
0.04	260685
0.06	173790
0.08	130342
0.1	104274
0.12	86895
0.14	74481
0.16	65171
0.18	57930
0.2	52137

2-16 The rate of heat generation per unit volume in the uranium rods is given. The total rate of heat generation in each rod is to be determined.



Assumptions Heat is generated uniformly in the uranium rods.

Analysis The total rate of heat generation in the rod is determined by multiplying the rate of heat generation per unit volume by the volume of the rod

$$\dot{G} = \dot{g}V_{\text{rod}} = \dot{g}(\pi D^2/4)L = (7\times10^7 \text{ W/m}^3)[\pi(0.05 \text{ m})^2/4](1 \text{ m}) = 1.374\times10^5 \text{ W} = \mathbf{137.4\ kW}$$

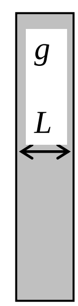
2-17 The variation of the absorption of solar energy in a solar pond with depth is given. A relation for the total rate of heat generation in a water layer at the top of the pond is to be determined.

Assumptions Absorption of solar radiation by water is modeled as heat generation.

Analysis The total rate of heat generation in a water layer of surface area A and thickness L at the top of the pond is determined by integration to be

$$\dot{G} = \int_V \dot{g}dV = \int_{x=0}^{L} \dot{g}_0 e^{-bx}(Adx) = A\dot{g}_0 \left. \frac{e^{-bx}}{-b} \right|_0^L = \frac{\mathbf{A\dot{g}_0(1-e^{-bL})}}{\mathbf{b}}$$

2-18 The rate of heat generation per unit volume in a stainless steel plate is given. The heat flux on the surface of the plate is to be determined.



Assumptions Heat is generated uniformly in steel plate.

Analysis We consider a unit surface area of 1 m^2. The total rate of heat generation in this section of the plate is

$$\dot{G} = \dot{g}V_{\text{plate}} = \dot{g}(A\times L) = (5\times10^6 \text{ W/m}^3)(1 \text{ m}^2)(0.03 \text{ m}) = 1.5\times10^5 \text{ W}$$

Noting that this heat will be dissipated from both sides of the plate, the heat flux on either surface of the plate becomes

$$\dot{q} = \frac{\dot{G}}{A_{\text{plate}}} = \frac{1.5\times10^5 \text{ W}}{2\times1 \text{ m}^2} = \mathbf{75{,}000\ W/m^2}$$

Heat Conduction Equation

2-19 The one-dimensional transient heat conduction equation for a plane wall with constant thermal conductivity and heat generation is $\frac{\partial^2 T}{\partial x^2} + \frac{\dot{g}}{k} = \frac{1}{\alpha}\frac{\partial T}{\partial t}$. Here T is the temperature, x is the space variable, g is the heat generation per unit volume, k is the thermal conductivity, α is the thermal diffusivity, and t is the time.

2-20 The one-dimensional transient heat conduction equation for a plane wall with constant thermal conductivity and heat generation is $\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right) + \frac{\dot{g}}{k} = \frac{1}{\alpha}\frac{\partial T}{\partial t}$. Here T is the temperature, r is the space variable, *g* is the heat generation per unit volume, *k* is the thermal conductivity, α is the thermal diffusivity, and *t* is the time.

2-21 We consider a thin element of thickness Δx in a large plane wall (see Fig. 2-13 in the text). The density of the wall is ρ, the specific heat is *C*, and the area of the wall normal to the direction of heat transfer is *A*. In the absence of any heat generation, an *energy balance* on this thin element of thickness Δx during a small time interval Δt can be expressed as

$$\dot{Q}_x - \dot{Q}_{x+\Delta x} = \frac{\Delta E_{\text{element}}}{\Delta t}$$

where

$$\Delta E_{\text{element}} = E_{t+\Delta t} - E_t = mC(T_{t+\Delta t} - T_t) = \rho CA\Delta x(T_{t+\Delta t} - T_t)$$

Substituting,

$$\dot{Q}_x - \dot{Q}_{x+\Delta x} = \rho CA\Delta x \frac{T_{t+\Delta t} - T_t}{\Delta t}$$

Dividing by $A\Delta x$ gives

$$-\frac{1}{A}\frac{\dot{Q}_{x+\Delta x} - \dot{Q}_x}{\Delta x} = \rho C \frac{T_{t+\Delta t} - T_t}{\Delta t}$$

Taking the limit as $\Delta x \to 0$ and $\Delta t \to 0$ yields

$$\frac{1}{A}\frac{\partial}{\partial x}\left(kA\frac{\partial T}{\partial x}\right) = \rho C \frac{\partial T}{\partial t}$$

since, from the definition of the derivative and Fourier's law of heat conduction,

$$\lim_{\Delta x \to 0} \frac{\dot{Q}_{x+\Delta x} - \dot{Q}_x}{\Delta x} = \frac{\partial Q}{\partial x} = \frac{\partial}{\partial x}\left(-kA\frac{\partial T}{\partial x}\right)$$

Noting that the area *A* of a plane wall is constant, the one-dimensional transient heat conduction equation in a plane wall with constant thermal conductivity *k* becomes

$$\frac{\partial^2 T}{\partial x^2} = \frac{1}{\alpha}\frac{\partial T}{\partial t}$$

where the property $\alpha = k / \rho C$ is the thermal diffusivity of the material.

2-22 We consider a thin cylindrical shell element of thickness Δr in a long cylinder (see Fig. 2-15 in the text). The density of the cylinder is ρ, the specific heat is *C*, and the length is *L*. The area of the cylinder normal to the direction of heat transfer at any location is $A = 2\pi rL$ where *r* is the value of the radius at that location. Note that the heat transfer area *A* depends on *r* in this case, and thus it varies with location. An *energy balance* on this thin cylindrical shell element of thickness Δr during a small time interval Δt can be expressed as

$$\dot{Q}_r - \dot{Q}_{r+\Delta r} + \dot{G}_{\text{element}} = \frac{\Delta E_{\text{element}}}{\Delta t}$$

where

$$\Delta E_{\text{element}} = E_{t+\Delta t} - E_t = mC(T_{t+\Delta t} - T_t) = \rho CA\Delta r(T_{t+\Delta t} - T_t)$$

$$\dot{G}_{\text{element}} = \dot{g}V_{\text{element}} = \dot{g}A\Delta r$$

Substituting,

$$\dot{Q}_r - \dot{Q}_{r+\Delta r} + \dot{g}A\Delta r = \rho CA\Delta r \frac{T_{t+\Delta t} - T_t}{\Delta t}$$

where $A = 2\pi rL$. Dividing the equation above by $A\Delta r$ gives

$$-\frac{1}{A}\frac{\dot{Q}_{r+\Delta r} - \dot{Q}_r}{\Delta r} + \dot{g} = \rho C \frac{T_{t+\Delta t} - T_t}{\Delta t}$$

Taking the limit as $\Delta r \to 0$ and $\Delta t \to 0$ yields

$$\frac{1}{A}\frac{\partial}{\partial r}\left(kA\frac{\partial T}{\partial r}\right) + \dot{g} = \rho C \frac{\partial T}{\partial t}$$

since, from the definition of the derivative and Fourier's law of heat conduction,

$$\lim_{\Delta r \to 0} \frac{\dot{Q}_{r+\Delta r} - \dot{Q}_r}{\Delta r} = \frac{\partial Q}{\partial r} = \frac{\partial}{\partial r}\left(-kA\frac{\partial T}{\partial r}\right)$$

Noting that the heat transfer area in this case is $A = 2\pi rL$ and the thermal conductivity is constant, the one-dimensional transient heat conduction equation in a cylinder becomes

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right) + \dot{g} = \frac{1}{\alpha}\frac{\partial T}{\partial t}$$

where $\alpha = k / \rho C$ is the thermal diffusivity of the material.

2-23 We consider a thin spherical shell element of thickness Δr in a sphere (see Fig. 2-17 in the text).. The density of the sphere is ρ, the specific heat is *C*, and the length is *L*. The area of the sphere normal to the direction of heat transfer at any location is $A = 4\pi r^2$ where *r* is the value of the radius at that location. Note that the heat transfer area *A* depends on *r* in this case, and thus it varies with location. When there is no heat generation, an *energy balance* on this thin spherical shell element of thickness Δr during a small time interval Δt can be expressed as

$$\dot{Q}_r - \dot{Q}_{r+\Delta r} = \frac{\Delta E_{\text{element}}}{\Delta t}$$

where

$$\Delta E_{\text{element}} = E_{t+\Delta t} - E_t = mC(T_{t+\Delta t} - T_t) = \rho CA\Delta r(T_{t+\Delta t} - T_t)$$

Substituting,

$$\dot{Q}_r - \dot{Q}_{r+\Delta r} + \dot{g}A\Delta r = \rho CA\Delta r\,\frac{T_{t+\Delta t} - T_t}{\Delta t}$$

where $A = 4\pi r^2$. Dividing the equation above by $A\Delta r$ gives

$$-\frac{1}{A}\frac{\dot{Q}_{r+\Delta r} - \dot{Q}_r}{\Delta r} = \rho C\frac{T_{t+\Delta t} - T_t}{\Delta t}$$

Taking the limit as $\Delta r \to 0$ and $\Delta t \to 0$ yields

$$\frac{1}{A}\frac{\partial}{\partial r}\left(kA\frac{\partial T}{\partial r}\right) = \rho C\frac{\partial T}{\partial t}$$

since, from the definition of the derivative and Fourier's law of heat conduction,

$$\lim_{\Delta r \to 0}\frac{\dot{Q}_{r+\Delta r} - \dot{Q}_r}{\Delta r} = \frac{\partial Q}{\partial r} = \frac{\partial}{\partial r}\left(-kA\frac{\partial T}{\partial r}\right)$$

Noting that the heat transfer area in this case is $A = 4\pi r^2$ and the thermal conductivity *k* is constant, the one-dimensional transient heat conduction equation in a sphere becomes

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial T}{\partial r}\right) = \frac{1}{\alpha}\frac{\partial T}{\partial t}$$

where $\alpha = k/\rho C$ is the thermal diffusivity of the material.

2-24 For a medium in which the heat conduction equation is given in its simplest by $\frac{\partial^2 T}{\partial x^2} = \frac{1}{\alpha}\frac{\partial T}{\partial t}$:

(*a*) Heat transfer is transient, (*b*) it is one-dimensional, (*c*) there is no heat generation, and (*d*) the thermal conductivity is constant.

2-25 For a medium in which the heat conduction equation is given in its simplest by $\frac{1}{r}\frac{d}{dr}\left(rk\frac{dT}{dr}\right) + \dot{g} = 0$:

(*a*) Heat transfer is steady, (*b*) it is one-dimensional, (*c*) there is heat generation, and (*d*) the thermal conductivity is variable.

2-26 For a medium in which the heat conduction equation is given by $\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial T}{\partial r}\right) = \frac{1}{\alpha}\frac{\partial T}{\partial t}$

(*a*) Heat transfer is transient, (*b*) it is one-dimensional, (*c*) there is no heat generation, and (*d*) the thermal conductivity is constant.

2-27 For a medium in which the heat conduction equation is given in its simplest by $r\frac{d^2T}{dr^2}+\frac{dT}{dr}=0$:

(*a*) Heat transfer is steady, (*b*) it is one-dimensional, (*c*) there is heat generation, and (*d*) the thermal conductivity is constant.

2-28 We consider a small rectangular element of length Δ*x*, width Δ*y*, and height Δ*z* = 1 (similar to the one in Fig. 2-21). The density of the body is ρ and the specific heat is *C*. Noting that heat conduction is two-dimensional and assuming no heat generation, an *energy balance* on this element during a small time interval Δ*t* can be expressed as

$$\begin{pmatrix}\text{Rate of heat}\\ \text{conduction at the}\\ \text{surfaces at } x \text{ and } y\end{pmatrix}-\begin{pmatrix}\text{Rate of heat conduction}\\ \text{at the surfaces at}\\ x+\Delta x \text{ and } y+\Delta y\end{pmatrix}=\begin{pmatrix}\text{Rate of change of}\\ \text{the energy content}\\ \text{of the element}\end{pmatrix}$$

or $$\dot{Q}_x+\dot{Q}_y-\dot{Q}_{x+\Delta x}-\dot{Q}_{y+\Delta y}=\frac{\Delta E_{\text{element}}}{\Delta t}$$

Noting that the volume of the element is $V_{\text{element}}=\Delta x\Delta y\Delta z=\Delta x\Delta y\times 1$, the change in the energy content of the element can be expressed as

$$\Delta E_{\text{element}}=E_{t+\Delta t}-E_t=mC(T_{t+\Delta t}-T_t)=\rho C\Delta x\Delta y(T_{t+\Delta t}-T_t)$$

Substituting, $$\dot{Q}_x+\dot{Q}_y-\dot{Q}_{x+\Delta x}-\dot{Q}_{y+\Delta y}=\rho C\Delta x\Delta y\frac{T_{t+\Delta t}-T_t}{\Delta t}$$

Dividing by $\Delta x\Delta y$ gives

$$-\frac{1}{\Delta y}\frac{\dot{Q}_{x+\Delta x}-\dot{Q}_x}{\Delta x}-\frac{1}{\Delta x}\frac{\dot{Q}_{y+\Delta y}-\dot{Q}_y}{\Delta y}=\rho C\frac{T_{t+\Delta t}-T_t}{\Delta t}$$

Taking the thermal conductivity *k* to be constant and noting that the heat transfer surface areas of the element for heat conduction in the *x* and *y* directions are $A_x=\Delta y\times 1$ and $A_y=\Delta x\times 1$, respectively, and taking the limit as $\Delta x,\ \Delta y,\text{ and }\Delta t\to 0$ yields

$$\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}=\frac{1}{\alpha}\frac{\partial T}{\partial t}$$

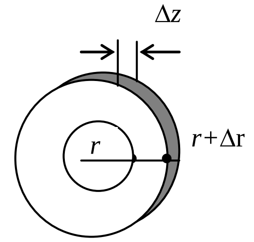
since, from the definition of the derivative and Fourier's law of heat conduction,

$$\lim_{\Delta x\to 0}\frac{1}{\Delta y\Delta z}\frac{\dot{Q}_{x+\Delta x}-\dot{Q}_x}{\Delta x}=\frac{1}{\Delta y\Delta z}\frac{\partial Q_x}{\partial x}=\frac{1}{\Delta y\Delta z}\frac{\partial}{\partial x}\left(-k\Delta y\Delta z\frac{\partial T}{\partial x}\right)=-\frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right)=-k\frac{\partial^2 T}{\partial x^2}$$

$$\lim_{\Delta y\to 0}\frac{1}{\Delta x\Delta z}\frac{\dot{Q}_{y+\Delta y}-\dot{Q}_y}{\Delta y}=\frac{1}{\Delta x\Delta z}\frac{\partial Q_y}{\partial y}=\frac{1}{\Delta x\Delta z}\frac{\partial}{\partial y}\left(-k\Delta x\Delta z\frac{\partial T}{\partial y}\right)=-\frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y}\right)=-k\frac{\partial^2 T}{\partial y^2}$$

Here the property $\alpha=k/\rho C$ is the thermal diffusivity of the material.

2-29 We consider a thin ring shaped volume element of width Δ*z* and thickness Δ*r* in a cylinder. The density of the cylinder is ρ and the specific heat is *C*. In general, an *energy balance* on this ring element during a small time interval Δt can be expressed as



$$(\dot{Q}_r - \dot{Q}_{r+\Delta r}) + (\dot{Q}_z - \dot{Q}_{z+\Delta z}) = \frac{\Delta E_{\text{element}}}{\Delta t}$$

But the change in the energy content of the element can be expressed as

$$\Delta E_{\text{element}} = E_{t+\Delta t} - E_t = mC(T_{t+\Delta t} - T_t) = \rho C(2\pi r\Delta r)\Delta z(T_{t+\Delta t} - T_t)$$

Substituting,

$$(\dot{Q}_r - \dot{Q}_{r+\Delta r}) + (\dot{Q}_z - \dot{Q}_{z+\Delta z}) = \rho C(2\pi r\Delta r)\Delta z \frac{T_{t+\Delta t} - T_t}{\Delta t}$$

Dividing the equation above by $(2\pi r\Delta r)\Delta z$ gives

$$-\frac{1}{2\pi r\Delta z}\frac{\dot{Q}_{r+\Delta r} - \dot{Q}_r}{\Delta r} - \frac{1}{2\pi r\Delta r}\frac{\dot{Q}_{z+\Delta z} - \dot{Q}_z}{\Delta z} = \rho C\frac{T_{t+\Delta t} - T_t}{\Delta t}$$

Noting that the heat transfer surface areas of the element for heat conduction in the *r* and *z* directions are $A_r = 2\pi r\Delta z$ and $A_z = 2\pi r\Delta r$, respectively, and taking the limit as $\Delta r,\ \Delta z$ and $\Delta t \to 0$ yields

$$\frac{1}{r}\frac{\partial}{\partial r}\left(kr\frac{\partial T}{\partial r}\right) + \frac{1}{r^2}\frac{\partial}{\partial \phi}\left(k\frac{\partial T}{\partial \phi}\right) + \frac{\partial}{\partial z}\left(k\frac{\partial T}{\partial z}\right) = \rho C\frac{\partial T}{\partial t}$$

since, from the definition of the derivative and Fourier's law of heat conduction,

$$\lim_{\Delta r\to 0}\frac{1}{2\pi r\Delta z}\frac{\dot{Q}_{r+\Delta r} - \dot{Q}_r}{\Delta r} = \frac{1}{2\pi r\Delta z}\frac{\partial Q}{\partial r} = \frac{1}{2\pi r\Delta z}\frac{\partial}{\partial r}\left(-k(2\pi r\Delta z)\frac{\partial T}{\partial r}\right) = -\frac{1}{r}\frac{\partial}{\partial r}\left(kr\frac{\partial T}{\partial r}\right)$$

$$\lim_{\Delta z\to 0}\frac{1}{2\pi r\Delta r}\frac{\dot{Q}_{z+\Delta z} - \dot{Q}_z}{\Delta z} = \frac{1}{2\pi r\Delta r}\frac{\partial Q_z}{\partial z} = \frac{1}{2\pi r\Delta r}\frac{\partial}{\partial z}\left(-k(2\pi r\Delta r)\frac{\partial T}{\partial z}\right) = -\frac{\partial}{\partial z}\left(k\frac{\partial T}{\partial z}\right)$$

For the case of constant thermal conductivity the equation above reduces to

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right) + \frac{\partial^2 T}{\partial z^2} = \frac{1}{\alpha}\frac{\partial T}{\partial t}$$

where $\alpha = k/\rho C$ is the thermal diffusivity of the material. For the case of steady heat conduction with no heat generation it reduces to

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right) + \frac{\partial^2 T}{\partial z^2} = 0$$

2-30 Consider a thin disk element of thickness Δz and diameter D in a long cylinder (Fig. P2-30). The density of the cylinder is ρ, the specific heat is C, and the area of the cylinder normal to the direction of heat transfer is $A = \pi D^2 / 4$, which is constant. An *energy balance* on this thin element of thickness Δz during a small time interval Δt can be expressed as

$$\left(\begin{array}{c}\text{Rate of heat}\\ \text{conduction at}\\ \text{the surface at } z\end{array}\right) - \left(\begin{array}{c}\text{Rate of heat}\\ \text{conduction at the}\\ \text{surface at } z+\Delta z\end{array}\right) + \left(\begin{array}{c}\text{Rate of heat}\\ \text{generation inside}\\ \text{the element}\end{array}\right) = \left(\begin{array}{c}\text{Rate of change of}\\ \text{the energy content}\\ \text{of the element}\end{array}\right)$$

or,

$$\dot{Q}_z - \dot{Q}_{z+\Delta z} + \dot{G}_{\text{element}} = \frac{\Delta E_{\text{element}}}{\Delta t}$$

But the change in the energy content of the element and the rate of heat generation within the element can be expressed as

$$\Delta E_{\text{element}} = E_{t+\Delta t} - E_t = mC(T_{t+\Delta t} - T_t) = \rho C A \Delta z (T_{t+\Delta t} - T_t)$$

and

$$\dot{G}_{\text{element}} = \dot{g} V_{\text{element}} = \dot{g} A \Delta z$$

Substituting,

$$\dot{Q}_z - \dot{Q}_{z+\Delta z} + \dot{g} A \Delta z = \rho C A \Delta z \frac{T_{t+\Delta t} - T_t}{\Delta t}$$

Dividing by $A\Delta z$ gives

$$-\frac{1}{A}\frac{\dot{Q}_{z+\Delta z} - \dot{Q}_z}{\Delta z} + \dot{g} = \rho C \frac{T_{t+\Delta t} - T_t}{\Delta t}$$

Taking the limit as $\Delta z \to 0$ and $\Delta t \to 0$ yields

$$\frac{1}{A}\frac{\partial}{\partial z}\left(kA\frac{\partial T}{\partial z}\right) + \dot{g} = \rho C \frac{\partial T}{\partial t}$$

since, from the definition of the derivative and Fourier's law of heat conduction,

$$\lim_{\Delta z \to 0} \frac{\dot{Q}_{z+\Delta z} - \dot{Q}_z}{\Delta z} = \frac{\partial Q}{\partial z} = \frac{\partial}{\partial z}\left(-kA\frac{\partial T}{\partial z}\right)$$

Noting that the area A and the thermal conductivity k are constant, the one-dimensional transient heat conduction equation in the axial direction in a long cylinder becomes

$$\frac{\partial^2 T}{\partial z^2} + \frac{\dot{g}}{k} = \frac{1}{\alpha}\frac{\partial T}{\partial t}$$

where the property $\alpha = k / \rho C$ is the thermal diffusivity of the material.

2-31 For a medium in which the heat conduction equation is given by $\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}=\frac{1}{\alpha}\frac{\partial T}{\partial t}$:

(*a*) Heat transfer is transient, (*b*) it is two-dimensional, (*c*) there is no heat generation, and (*d*) the thermal conductivity is constant.

2-32 For a medium in which the heat conduction equation is given by $\frac{1}{r}\frac{\partial}{\partial r}\left(kr\frac{\partial T}{\partial r}\right)+\frac{\partial}{\partial z}\left(k\frac{\partial T}{\partial z}\right)+\dot{g}=0$:

(*a*) Heat transfer is steady, (*b*) it is two-dimensional, (*c*) there is heat generation, and (*d*) the thermal conductivity is variable.

2-33 For a medium in which the heat conduction equation is given by

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial T}{\partial r}\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial^2 T}{\partial \phi^2}=\frac{1}{\alpha}\frac{\partial T}{\partial t}:$$

(*a*) Heat transfer is transient, (*b*) it is two-dimensional, (*c*) there is no heat generation, and (*d*) the thermal conductivity is constant.

Boundary and Initial Conditions; Formulation of Heat Conduction Problems

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2-34C The mathematical expressions of the thermal conditions at the boundaries are called the **boundary conditions.** To describe a heat transfer problem completely, *two boundary conditions* must be given for *each direction* of the coordinate system along which heat transfer is significant. Therefore, we need to specify four boundary conditions for two-dimensional problems.

2-35C The mathematical expression for the temperature distribution of the medium initially is called the **initial condition**. We need only one initial condition for a heat conduction problem regardless of the dimension since the conduction equation is first order in time (it involves the first derivative of temperature with respect to time). Therefore, we need only 1 initial condition for a two-dimensional problem.

2-36C A heat transfer problem that is symmetric about a plane, line, or point is said to have thermal symmetry about that plane, line, or point. The thermal symmetry boundary condition is a mathematical expression of this thermal symmetry. It is equivalent to *insulation* or *zero heat flux* boundary condition, and is expressed at a point x_0 as $\partial T(x_0,t)/\partial x=0$.

2-37C The boundary condition at a perfectly insulated surface (at x = 0, for example) can be expressed as

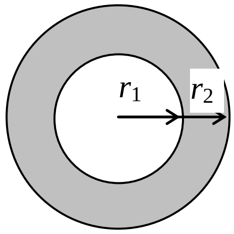
$$-k\frac{\partial T(0,t)}{\partial x}=0 \qquad \text{or} \qquad \frac{\partial T(0,t)}{\partial x}=0$$

which indicates zero heat flux.

2-38C Yes, the temperature profile in a medium must be perpendicular to an insulated surface since the slope $\partial T/\partial x=0$ at that surface.

2-39C We try to avoid the radiation boundary condition in heat transfer analysis because it is a non-linear expression that causes mathematical difficulties while solving the problem; often making it impossible to obtain analytical solutions.

2-40 A spherical container of inner radius r_1, outer radius r_2, and thermal conductivity k is given. The boundary condition on the inner surface of the container for steady one-dimensional conduction is to be expressed for the following cases:



(*a*) Specified temperature of 50°C: $T(r_1) = 50°\text{C}$

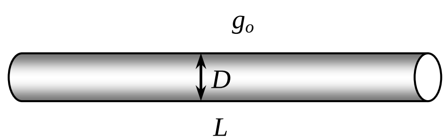
(*b*) Specified heat flux of 30 W/m^2 towards the center: $k\dfrac{dT(r_1)}{dr} = 30\ \text{W/m}^2$

(*c*) Convection to a medium at T_∞ with a heat transfer coefficient of *h*: $k\dfrac{dT(r_1)}{dr} = h[T(r_1) - T_\infty]$

2-41 Heat is generated in a long wire of radius r_0 covered with a plastic insulation layer at a constant rate of $\dot{g}_0$. The heat flux boundary condition at the interface (radius r_0) in terms of the heat generated is to be expressed. The total heat generated in the wire and the heat flux at the interface are

$$\dot{G} = \dot{g}_0 V_{\text{wire}} = \dot{g}_0(\pi r_0^2 L)$$

$$\dot{q}_s = \frac{\dot{Q}_s}{A} = \frac{\dot{G}}{A} = \frac{\dot{g}_0(\pi r_0^2 L)}{(2\pi r_0)L} = \frac{\dot{g}_0 r_0}{2}$$

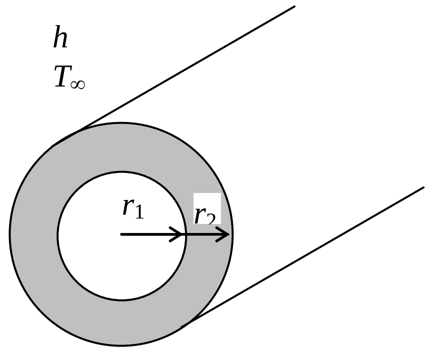


Assuming steady one-dimensional conduction in the radial direction, the heat flux boundary condition can be expressed as

$$-k\frac{dT(r_0)}{dr} = \frac{\dot{g}_0 r_0}{2}$$

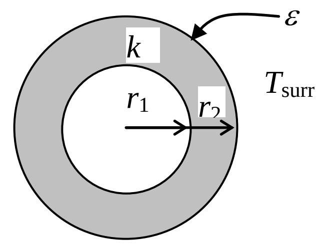
2-42 A long pipe of inner radius r_1, outer radius r_2, and thermal conductivity k is considered. The outer surface of the pipe is subjected to convection to a medium at T_∞ with a heat transfer coefficient of *h*. Assuming steady one-dimensional conduction in the radial direction, the convection boundary condition on the outer surface of the pipe can be expressed as

$$-k\frac{dT(r_2)}{dr} = h[T(r_2) - T_\infty]$$



2-43 A spherical shell of inner radius r_1, outer radius r_2, and thermal conductivity k is considered. The outer surface of the shell is subjected to radiation to surrounding surfaces at T_{surr}. Assuming no convection and steady one-dimensional conduction in the radial direction, the radiation boundary condition on the outer surface of the shell can be expressed as

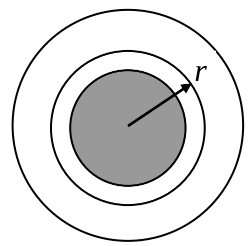
$$-k\frac{dT(r_2)}{dr} = \varepsilon\sigma\ T(r_2)^4 - T_{\text{surr}}^4$$



2-44 A spherical container consists of two spherical layers A and B that are at perfect contact. The radius of the interface is r_o. Assuming transient one-dimensional conduction in the radial direction, the boundary conditions at the interface can be expressed as

$$T_A(r_0,t) = T_B(r_0,t)$$

and $$-k_A\frac{\partial T_A(r_0,t)}{\partial x} = -k_B\frac{\partial T_B(r_0,t)}{\partial x}$$



2-45 Heat conduction through the bottom section of a steel pan that is used to boil water on top of an electric range is considered (Fig. P2-45). Assuming constant thermal conductivity and one-dimensional heat transfer, the mathematical formulation (the differential equation and the boundary conditions) of this heat conduction problem is to be obtained for steady operation.

Assumptions **1** Heat transfer is given to be steady and one-dimensional. **2** Thermal conductivity is given to be constant. **3** There is no heat generation in the medium. **4** The top surface at $x = L$ is subjected to convection and the bottom surface at $x = 0$ is subjected to uniform heat flux.

Analysis The heat flux at the bottom of the pan is

$$\dot{q}_s = \frac{\dot{Q}_s}{A_s} = \frac{\dot{G}}{\pi D^2/4} = \frac{0.85\times(1000\text{ W})}{\pi(0.20\text{ m})^2/4} = 27{,}056\text{ W/m}^2$$

Then the differential equation and the boundary conditions

for this heat conduction problem can be expressed as

$$\frac{d^2T}{dx^2} = 0$$

$$-k\frac{dT(0)}{dr} = \dot{q}_s = 27{,}056\text{ W/m}^2$$

$$-k\frac{dT(L)}{dr} = h[T(L) - T_\infty]$$

2-46E A 1.5-kW resistance heater wire is used for space heating. Assuming constant thermal conductivity and one-dimensional heat transfer, the mathematical formulation (the differential equation and the boundary conditions) of this heat conduction problem is to be obtained for steady operation.

Assumptions **1** Heat transfer is given to be steady and one-dimensional. **2** Thermal conductivity is given to be constant. **3** Heat is generated uniformly in the wire.

Analysis The heat flux at the surface of the wire is

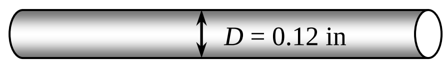
$$\dot{q}_s = \frac{\dot{Q}_s}{A_s} = \frac{\dot{G}}{2\pi r_0 L} = \frac{1200 \text{ W}}{2\pi(0.06 \text{ in})(15 \text{ in})} = 212.2 \text{ W/in}^2$$

Noting that there is thermal symmetry about the center line and there is uniform heat flux at the outer surface, the differential equation and the boundary conditions for this heat conduction problem can be expressed as

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{dT}{dr}\right) + \frac{\dot{g}}{k} = 0$$

$$\frac{dT(0)}{dr} = 0$$

$$-k\frac{dT(r_0)}{dr} = \dot{q}_s = 212.2 \text{ W/in}^2$$



2-47 Heat conduction through the bottom section of an aluminum pan that is used to cook stew on top of an electric range is considered (Fig. P2-47). Assuming variable thermal conductivity and one-dimensional heat transfer, the mathematical formulation (the differential equation and the boundary conditions) of this heat conduction problem is to be obtained for steady operation.

Assumptions **1** Heat transfer is given to be steady and one-dimensional. **2** Thermal conductivity is given to be variable. **3** There is no heat generation in the medium. **4** The top surface at $x = L$ is subjected to specified temperature and the bottom surface at $x = 0$ is subjected to uniform heat flux.

Analysis The heat flux at the bottom of the pan is

$$\dot{q}_s = \frac{\dot{Q}_s}{A_s} = \frac{\dot{G}}{\pi D^2/4} = \frac{0.90 \times (900 \text{ W})}{\pi(0.18 \text{ m})^2/4} = 31{,}831 \text{ W/m}^2$$

Then the differential equation and the boundary conditions

for this heat conduction problem can be expressed as

$$\frac{d}{dx}\left(k\frac{dT}{dx}\right) = 0$$

$$-k\frac{dT(0)}{dr} = \dot{q}_s = 31{,}831 \text{ W/m}^2$$

$$T(L) = T_L = 108°\text{C}$$

2-48 Water flows through a pipe whose outer surface is wrapped with a thin electric heater that consumes 300 W per m length of the pipe. The exposed surface of the heater is heavily insulated so that the entire heat generated in the heater is transferred to the pipe. Heat is transferred from the inner surface of the pipe to the water by convection. Assuming constant thermal conductivity and one-dimensional heat transfer, the mathematical formulation (the differential equation and the boundary conditions) of the heat conduction in the pipe is to be obtained for steady operation.

Assumptions **1** Heat transfer is given to be steady and one-dimensional. **2** Thermal conductivity is given to be constant. **3** There is no heat generation in the medium. **4** The outer surface at $r = r_2$ is subjected to uniform heat flux and the inner surface at $r = r_1$ is subjected to convection.

Analysis The heat flux at the outer surface of the pipe is

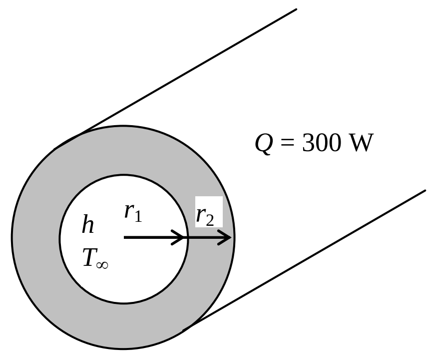
$$\dot{q}_s = \frac{\dot{Q}_s}{A_s} = \frac{\dot{Q}_s}{2\pi r_2 L} = \frac{300 \text{ W}}{2\pi(0.065 \text{ cm})(1 \text{ m})} = 734.6 \text{ W/m}^2$$

Noting that there is thermal symmetry about the center line and there is uniform heat flux at the outer surface, the differential equation and the boundary conditions for this heat conduction problem can be expressed as

$$\frac{d}{dr}\left(r\frac{dT}{dr}\right) = 0$$

$$k\frac{dT(r_1)}{dr} = h[T(r_i) - T_\infty]$$

$$k\frac{dT(r_2)}{dr} = \dot{q}_s = 734.6 \text{ W/m}^2$$



2-49 A spherical metal ball that is heated in an oven to a temperature of T_i throughout is dropped into a large body of water at T_∞ where it is cooled by convection. Assuming constant thermal conductivity and transient one-dimensional heat transfer, the mathematical formulation (the differential equation and the boundary and initial conditions) of this heat conduction problem is to be obtained.

Assumptions **1** Heat transfer is given to be transient and one-dimensional. **2** Thermal conductivity is given to be constant. **3** There is no heat generation in the medium. **4** The outer surface at $r = r_0$ is subjected to convection.

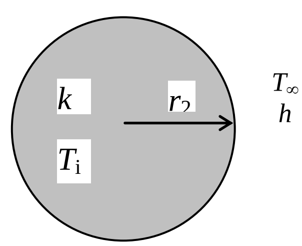
Analysis Noting that there is thermal symmetry about the midpoint and convection at the outer surface, the differential equation and the boundary conditions for this heat conduction problem can be expressed as

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial T}{\partial r}\right) = \frac{1}{\alpha}\frac{\partial T}{\partial t}$$

$$\frac{\partial T(0,t)}{\partial r} = 0$$

$$-k\frac{\partial T(r_0,t)}{\partial r} = h[T(r_0) - T_\infty]$$

$$T(r,0) = T_i$$



2-50 A spherical metal ball that is heated in an oven to a temperature of T_i throughout is allowed to cool in ambient air at T_∞ by convection and radiation. Assuming constant thermal conductivity and transient one-dimensional heat transfer, the mathematical formulation (the differential equation and the boundary and initial conditions) of this heat conduction problem is to be obtained.

Assumptions **1** Heat transfer is given to be transient and one-dimensional. **2** Thermal conductivity is given to be variable. **3** There is no heat generation in the medium. **4** The outer surface at $r = r_0$ is subjected to convection and radiation.

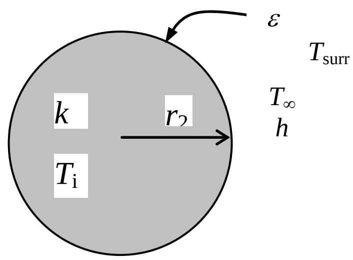
Analysis Noting that there is thermal symmetry about the midpoint and convection and radiation at the outer surface and expressing all temperatures in Rankine, the differential equation and the boundary conditions for this heat conduction problem can be expressed as

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(kr^2\frac{\partial T}{\partial r}\right) = \rho C\frac{\partial T}{\partial t}$$

$$\frac{\partial T(0,t)}{\partial r} = 0$$

$$-k\frac{\partial T(r_0,t)}{\partial r} = h[T(r_0) - T_\infty] + \varepsilon\sigma[T(r_0)^4 - T_{surr}^4]$$

$$T(r,0) = T_i$$



2-51 The outer surface of the North wall of a house exchanges heat with both convection and radiation., while the interior surface is subjected to convection only. Assuming the heat transfer through the wall to be steady and one-dimensional, the mathematical formulation (the differential equation and the boundary and initial conditions) of this heat conduction problem is to be obtained.

Assumptions **1** Heat transfer is given to be steady and one-dimensional. **2** Thermal conductivity is given to be constant. **3** There is no heat generation in the medium. **4** The outer surface at $x = L$ is subjected to convection and radiation while the inner surface at $x = 0$ is subjected to convection only.

Analysis Expressing all the temperatures in Kelvin, the differential equation and

the boundary conditions for this heat conduction problem can be expressed as

$$\frac{d^2T}{dx^2} = 0$$

$$-k\frac{dT(0)}{dx} = h_1[T_{\infty 1} - T(0)]$$

$$-k\frac{dT(L)}{dx} = h_1[T(L) - T_{\infty 2}] + \varepsilon_2\sigma\ T(L)^4 - T_{sky}^4$$