Variable Thermal Conductivity

2-94C During steady one-dimensional heat conduction in a plane wall, long cylinder, and sphere with constant thermal conductivity and no heat generation, the temperature in only the *plane wall* will vary linearly.

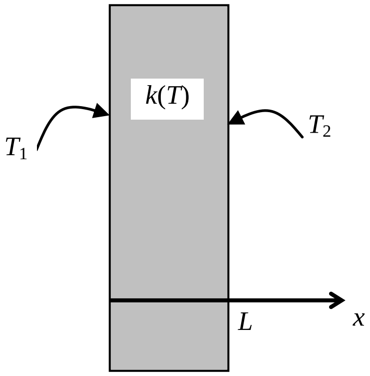
2-95C The thermal conductivity of a medium, in general, varies with temperature.

2-96C During steady one-dimensional heat conduction in a plane wall in which the thermal conductivity varies linearly, the error involved in heat transfer calculation by assuming constant thermal conductivity at the average temperature is (*a*) *none*.

2-97C No, the temperature variation in a plain wall will not be linear when the thermal conductivity varies with temperature.

2-98C Yes, when the thermal conductivity of a medium varies linearly with temperature, the average thermal conductivity is always equivalent to the conductivity value at the average temperature.

2-99 A plate with variable conductivity is subjected to specified temperatures on both sides. The rate of heat transfer through the plate is to be determined.



Assumptions **1** Heat transfer is given to be steady and one-dimensional. **2** Thermal conductivity varies quadratically. **3** There is no heat generation.

Properties The thermal conductivity is given to be $k(T) = k_0(1+\beta T^2)$.

Analysis When the variation of thermal conductivity with temperature *k*(*T*) is known, the average value of the thermal conductivity in the temperature range between T_1 and T_2 can be determined from

$$k_{\text{ave}} = \frac{\int_{T_1}^{T_2} k(T)dT}{T_2 - T_1} = \frac{\int_{T_1}^{T_2} k_0(1+\beta T^2)dT}{T_2 - T_1} = \frac{k_0\left(T + \dfrac{\beta}{3}T^3\right)_{T_1}^{T_2}}{T_2 - T_1} = \frac{k_0\left[(T_2 - T_1) + \dfrac{\beta}{3}\left(T_2^3 - T_1^3\right)\right]}{T_2 - T_1}$$

$$= k_0\left[1 + \frac{\beta}{3}\left(T_2^2 + T_1T_2 + T_1^2\right)\right]$$

This relation is based on the requirement that the rate of heat transfer through a medium with constant average thermal conductivity k_{ave} equals the rate of heat transfer through the same medium with variable conductivity *k*(*T*). Then the rate of heat conduction through the plate can be determined to be

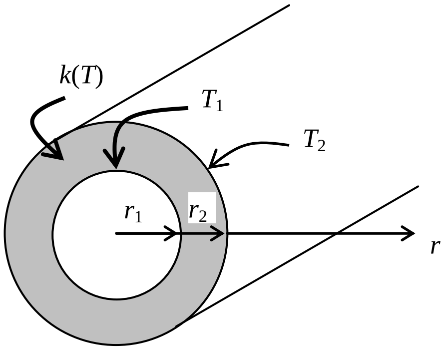
$$\dot{Q} = k_{\text{ave}} A \frac{T_1 - T_2}{L} = k_0\left[1 + \frac{\beta}{3}\left(T_2^2 + T_1T_2 + T_1^2\right)\right] A \frac{T_1 - T_2}{L}$$

Discussion We would obtain the same result if we substituted the given *k*(*T*) relation into the second part of Eq. 2-76, and performed the indicated integration.

2-100 A cylindrical shell with variable conductivity is subjected to specified temperatures on both sides. The variation of temperature and the rate of heat transfer through the shell are to be determined.

Assumptions **1** Heat transfer is given to be steady and one-dimensional. **2** Thermal conductivity varies linearly. **3** There is no heat generation.

Properties The thermal conductivity is given to be $k(T) = k_0(1+\beta T)$.



Solution (*a*) The rate of heat transfer through the shell is expressed as

$$\dot{Q}_{\text{cylinder}} = 2\pi k_{\text{ave}} L \frac{T_1 - T_2}{\ln(r_2 / r_1)}$$

where L is the length of the cylinder, r_1 is the inner radius, and r_2 is the outer radius, and

$$k_{\text{ave}} = k(T_{\text{ave}}) = k_0\left(1+\beta\frac{T_2+T_1}{2}\right)$$

is the average thermal conductivity.

(*b*) To determine the temperature distribution in the shell, we begin with the Fourier's law of heat conduction expressed as

$$\dot{Q} = -k(T)A\frac{dT}{dr}$$

where the rate of conduction heat transfer $\dot{Q}$ is constant and the heat conduction area $A = 2\pi rL$ is variable. Separating the variables in the above equation and integrating from $r = r_1$ where $T(r_1) = T_1$ to any r where $T(r) = T$, we get

$$\dot{Q}\int_{r_1}^{r}\frac{dr}{r} = -2\pi L\int_{T_1}^{T}k(T)dT$$

Substituting $k(T) = k_0(1+\beta T)$ and performing the integrations gives

$$\dot{Q}\ln\frac{r}{r_1} = -2\pi L k_0[(T-T_1)+\beta(T^2-T_1^2)/2]$$

Substituting the $\dot{Q}$ expression from part (*a*) and rearranging give

$$T^2 + \frac{2}{\beta}T + \frac{2k_{\text{ave}}}{\beta k_0}\frac{\ln(r/r_1)}{\ln(r_2/r_1)}(T_1 - T_2) - T_1^2 - \frac{2}{\beta}T_1 = 0$$

which is a *quadratic* equation in the unknown temperature *T*. Using the quadratic formula, the temperature distribution *T*(*r*) in the cylindrical shell is determined to be

$$T(r) = -\frac{1}{\beta} \pm \sqrt{\frac{1}{\beta^2} - \frac{2k_{\text{ave}}}{\beta k_0}\frac{\ln(r/r_1)}{\ln(r_2/r_1)}(T_1 - T_2) + T_1^2 + \frac{2}{\beta}T_1}$$

Discussion The proper sign of the square root term (+ or -) is determined from the requirement that the temperature at any point within the medium must remain between T_1 and T_2.

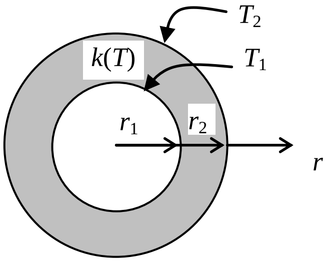
2-101 A spherical shell with variable conductivity is subjected to specified temperatures on both sides. The variation of temperature and the rate of heat transfer through the shell are to be determined.

Assumptions **1** Heat transfer is given to be steady and one-dimensional. **2** Thermal conductivity varies linearly. **3** There is no heat generation.

Properties The thermal conductivity is given to be $k(T) = k_0(1+\beta T)$.

Solution (*a*) The rate of heat transfer through the shell is expressed as

$$\dot{Q}_{\text{sphere}} = 4\pi k_{\text{ave}} r_1 r_2 \frac{T_1 - T_2}{r_2 - r_1}$$



where r_1 is the inner radius, r_2 is the outer radius, and

$$k_{\text{ave}} = k(T_{\text{ave}}) = k_0\left(1+\beta\frac{T_2+T_1}{2}\right)$$

is the average thermal conductivity.

(*b*) To determine the temperature distribution in the shell, we begin with the Fourier's law of heat conduction expressed as

$$\dot{Q} = -k(T)A\frac{dT}{dr}$$

where the rate of conduction heat transfer $\dot{Q}$ is constant and the heat conduction area $A = 4\pi r^2$ is variable. Separating the variables in the above equation and integrating from $r = r_1$ where $T(r_1) = T_1$ to any r where $T(r) = T$, we get

$$\dot{Q}\int_{r_1}^{r}\frac{dr}{r^2} = -4\pi\int_{T_1}^{T}k(T)dT$$

Substituting $k(T) = k_0(1+\beta T)$ and performing the integrations gives

$$\dot{Q}\left(\frac{1}{r_1}-\frac{1}{r}\right) = -4\pi k_0[(T-T_1)+\beta(T^2-T_1^2)/2]$$

Substituting the $\dot{Q}$ expression from part (*a*) and rearranging give

$$T^2 + \frac{2}{\beta}T + \frac{2k_{\text{ave}}}{\beta k_0}\frac{r_2(r-r_1)}{r(r_2-r_1)}(T_1-T_2) - T_1^2 - \frac{2}{\beta}T_1 = 0$$

which is a *quadratic* equation in the unknown temperature *T*. Using the quadratic formula, the temperature distribution *T*(*r*) in the cylindrical shell is determined to be

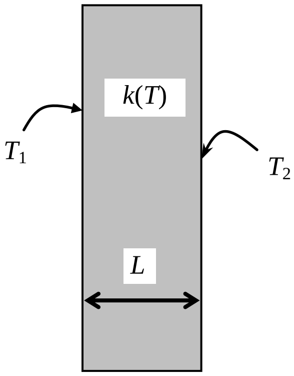
$$T(r) = -\frac{1}{\beta} \pm \sqrt{\frac{1}{\beta^2} - \frac{2k_{\text{ave}}}{\beta k_0}\frac{r_2(r-r_1)}{r(r_2-r_1)}(T_1-T_2) + T_1^2 + \frac{2}{\beta}T_1}$$

Discussion The proper sign of the square root term (+ or -) is determined from the requirement that the temperature at any point within the medium must remain between T_1 and T_2.

2-102 A plate with variable conductivity is subjected to specified temperatures on both sides. The rate of heat transfer through the plate is to be determined.

Assumptions **1** Heat transfer is given to be steady and one-dimensional. **2** Thermal conductivity varies linearly. **3** There is no heat generation.

Properties The thermal conductivity is given to be $k(T) = k_0(1+\beta T)$.



Analysis The average thermal conductivity of the medium in this case is simply the conductivity value at the average temperature since the thermal conductivity varies linearly with temperature, and is determined to be

$$\begin{aligned} k_{\text{ave}} &= k(T_{\text{ave}}) = k_0\left(1+\beta\frac{T_2+T_1}{2}\right) \\ &= (25\ \text{W/m}\cdot\text{K})\left(1+(8.7\times10^{-4}\ \text{K}^{-1})\frac{(500+350)\,\text{K}}{2}\right) \\ &= 34.24\ \text{W/m}\cdot\text{K} \end{aligned}$$

Then the rate of heat conduction through the plate becomes

$$\dot{Q} = k_{\text{ave}}A\frac{T_1-T_2}{L} = (34.24\ \text{W/m}\cdot\text{K})(1.5\,\text{m}\times0.6\,\text{m})\frac{(500-350)\text{K}}{0.15\,\text{m}} = \mathbf{30{,}820\,W}$$

Discussion We would obtain the same result if we substituted the given $k(T)$ relation into the second part of Eq, 2-76, and performed the indicated integration.

2-103

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"GIVEN"
A=1.5*0.6 "[m^2]"
L=0.15 "[m]"
"T_1=500 [K], parameter to be varied"
T_2=350 "[K]"
k_0=25 "[W/m-K]"
beta=8.7E-4 "[1/K]"

"ANALYSIS"
k=k_0*(1+beta*T)
T=1/2*(T_1+T_2)
Q_dot=k*A*(T_1-T_2)/L
```

T_1 [W]	Q [W]
400	9947
425	15043
450	20220
475	25479
500	30819
525	36241
550	41745
575	47330
600	52997
625	58745
650	64575
675	70486
700	76479

Special Topic: Review of Differential equations

2-104C We utilize appropriate simplifying assumptions when deriving differential equations to obtain an equation that we can deal with and solve.

2-105C A **variable** is a quantity which may assume various values during a study. A variable whose value can be changed arbitrarily is called an **independent variable** (or argument). A variable whose value depends on the value of other variables and thus cannot be varied independently is called a **dependent variable** (or a function).

2-106C A differential equation may involve more than one dependent or independent variable. For example, the equation $\frac{\partial^2 T(x,t)}{\partial x^2}+\frac{\dot{g}}{k}=\frac{1}{\alpha}\frac{\partial T(x,t)}{\partial t}$ has one dependent (*T*) and 2 independent variables (*x* and *t*). the equation $\frac{\partial^2 T(x,t)}{\partial x^2}+\frac{\partial W(x,t)}{\partial x}=\frac{1}{\alpha}\frac{\partial T(x,t)}{\partial t}+\frac{1}{\alpha}\frac{\partial W(x,t)}{\partial t}$ has 2 dependent (*T* and *W*) and 2 independent variables (*x* and *t*).

2-107C Geometrically, the **derivative** of a function *y*(*x*) at a point represents the *slope* of the tangent line to the graph of the function at that point. The derivative of a function that depends on two or more independent variables with respect to one variable while holding the other variables constant is called the partial derivative. Ordinary and partial derivatives are equivalent for functions that depend on a single independent variable.

2-108C The order of a derivative represents the number of times a function is differentiated, whereas the degree of a derivative represents how many times a derivative is multiplied by itself. For example, y''' is the third order derivative of *y*, whereas $(y')^3$ is the third degree of the first derivative of y.

2-109C For a function $f(x,y)$, the partial derivative $\partial f/\partial x$ will be equal to the ordinary derivative df/dx when *f* does not depend on *y* or this dependence is negligible.

2-110C For a function $f(x)$, the derivative df/dx does not have to be a function of *x*. The derivative will be a constant when the *f* is a linear function of *x*.

2-111C Integration is the inverse of derivation. Derivation increases the order of a derivative by one, integration reduces it by one.

2-112C A differential equation involves derivatives, an algebraic equation does not.

2-113C A differential equation that involves only ordinary derivatives is called an ordinary differential equation, and a differential equation that involves partial derivatives is called a partial differential equation.

2-114C The order of a differential equation is the order of the highest order derivative in the equation.

2-115C A differential equation is said to be **linear** if the dependent variable and all of its derivatives are of the first degree, and their coefficients depend on the independent variable only. In other words, a differential equation is linear if it can be written in a form which does not involve (1) any powers of the dependent variable or its derivatives such as y^3 or $(y')^2$, (2) any products of the dependent variable or its derivatives such as yy' or $y'y'''$, and (3) any other nonlinear functions of the dependent variable such as $\sin y$ or e^y. Otherwise, it is **nonlinear**.

2-116C A linear homogeneous differential equation of order *n* is expressed in the most general form as

$$y^{(n)} + f_1(x)y^{(n-1)} + \cdots + f_{n-1}(x)y' + f_n(x)y = 0$$

Each term in a linear homogeneous equation contains the dependent variable or one of its derivatives after the equation is cleared of any common factors. The equation $y'' - 4x^2 y = 0$ is linear and homogeneous since each term is linear in *y,* and contains the dependent variable or one of its derivatives.

2-117C A differential equation is said to have **constant coefficients** if the coefficients of all the terms which involve the dependent variable or its derivatives are constants. If, after cleared of any common factors, any of the terms with the dependent variable or its derivatives involve the independent variable as a coefficient, that equation is said to have **variable coefficients** The equation $y'' - 4x^2 y = 0$ has variable coefficients whereas the equation $y'' - 4y = 0$ has constant coefficients.

2-118C A linear differential equation that involves a single term with the derivatives can be solved by direct integration.

2-119C The general solution of a 3rd order linear and homogeneous differential equation will involve 3 arbitrary constants.

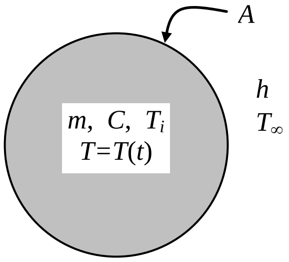
Review Problems

2-120 A small hot metal object is allowed to cool in an environment by convection. The differential equation that describes the variation of temperature of the ball with time is to be derived.

Assumptions **1** The temperature of the metal object changes uniformly with time during cooling so that $T = T(t)$. **2** The density, specific heat, and thermal conductivity of the body are constant. **3** There is no heat generation.

Analysis Consider a body of arbitrary shape of mass *m*, volume *V*, surface area *A*, density ρ, and specific heat C_p initially at a uniform temperature T_i . At time $t = 0$, the body is placed into a medium at temperature T_∞, and heat transfer takes place between the body and its environment with a heat transfer coefficient *h*.

During a differential time interval *dt*, the temperature of the body rises by a differential amount *dT*. Noting that the temperature changes with time only, an energy balance of the solid for the time interval *dt* can be expressed as



$$\begin{pmatrix}\text{Heat transfer from the body}\\ \text{during } dt\end{pmatrix} = \begin{pmatrix}\text{The decrease in the energy}\\ \text{of the body during } dt\end{pmatrix}$$

or $$hA_s(T - T_\infty)dt = mC_p(-dT)$$

Noting that $m = \rho V$ and $dT = d(T - T_\infty)$ since T_∞ = constant, the equation above can be rearranged as

$$\frac{d(T - T_\infty)}{T - T_\infty} = -\frac{hA_s}{\rho VC_p}dt$$

which is the desired differential equation.

2-121 A long rectangular bar is initially at a uniform temperature of T_i. The surfaces of the bar at $x = 0$ and $y = 0$ are insulated while heat is lost from the other two surfaces by convection. The mathematical formulation of this heat conduction problem is to be expressed for transient two-dimensional heat transfer with no heat generation.

Assumptions **1** Heat transfer is transient and two-dimensional. **2** Thermal conductivity is constant. **3** There is no heat generation.

Analysis The differential equation and the boundary conditions for this heat conduction problem can be expressed as

$$\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}=\frac{1}{\alpha}\frac{\partial T}{\partial t}$$

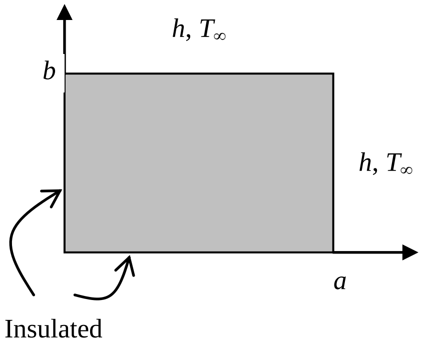
$$\frac{\partial T(x,0,t)}{\partial x}=0$$

$$\frac{\partial T(0,y,t)}{\partial x}=0$$

$$-k\frac{\partial T(a,y,t)}{\partial x}=h[T(a,y,t)-T_\infty]$$

$$-k\frac{\partial T(x,b,t)}{\partial x}=h[T(x,b,t)-T_\infty]$$

$$T(x,y,0)=T_i$$



2-122 Heat is generated at a constant rate in a short cylinder. Heat is lost from the cylindrical surface at $r = r_0$ by convection to the surrounding medium at temperature T_∞ with a heat transfer coefficient of h. The bottom surface of the cylinder at $r = 0$ is insulated, the top surface at $z = H$ is subjected to uniform heat flux $\dot{q}_h$, and the cylindrical surface at $r = r_0$ is subjected to convection. The mathematical formulation of this problem is to be expressed for steady two-dimensional heat transfer.

Assumptions **1** Heat transfer is given to be steady and two-dimensional. **2** Thermal conductivity is constant. **3** Heat is generated uniformly.

Analysis The differential equation and the boundary conditions for this heat conduction problem can be expressed as

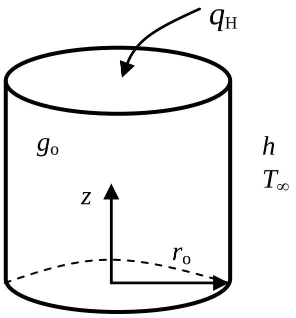
$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right)+\frac{\partial^2 T}{\partial z^2}+\frac{\dot{g}}{k}=0$$

$$\frac{\partial T(r,0)}{\partial z}=0$$

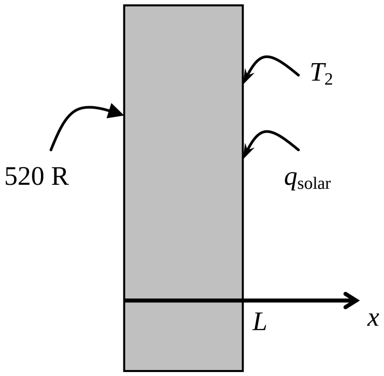
$$k\frac{\partial T(r,h)}{\partial z}=\dot{q}_H$$

$$\frac{\partial T(0,z)}{\partial r}=0$$

$$-k\frac{\partial T(r_0,z)}{\partial r}=h[T(r_0,z)-T_\infty]$$



2-123E A large plane wall is subjected to a specified temperature on the left (inner) surface and solar radiation and heat loss by radiation to space on the right (outer) surface. The temperature of the right surface of the wall and the rate of heat transfer are to be determined when steady operating conditions are reached.



Assumptions **1** Steady operating conditions are reached. **2** Heat transfer is one-dimensional since the wall is large relative to its thickness, and the thermal conditions on both sides of the wall are uniform. **3** Thermal properties are constant. **4** There is no heat generation in the wall.

Properties The properties of the plate are given to be k = 1.2 Btu/h·ft·°F and ε = 0.80, and $\alpha_s = 0.45$.

Analysis In steady operation, heat conduction through the wall must be equal to net heat transfer from the outer surface. Therefore, taking the outer surface temperature of the plate to be T_2 (absolute, in R),

$$kA_s \frac{T_1 - T_2}{L} = \varepsilon\sigma A_s T_2^4 - \alpha_s A_s \dot{q}_{\text{solar}}$$

Canceling the area A and substituting the known quantities,

$$(1.2\ \text{Btu/h}\cdot\text{ft}\cdot{}^\circ\text{F})\frac{(520\ \text{R}) - T_2}{0.5\ \text{ft}} = 0.8(0.1714\times 10^{-8}\ \text{Btu/h}\cdot\text{ft}^2\cdot\text{R}^4)T_2^4 - 0.45(300\ \text{Btu/h}\cdot\text{ft}^2)$$

Solving for T_2 gives the outer surface temperature to be T_2 = **530.9 R**

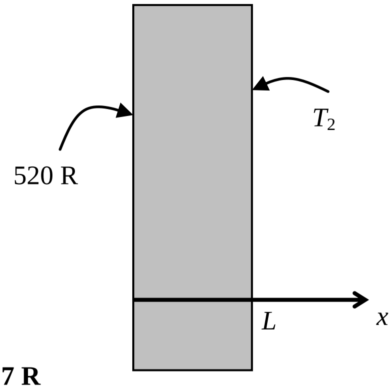
Then the rate of heat transfer through the wall becomes

$$\dot{q} = k\frac{T_1 - T_2}{L} = (1.2\ \text{Btu/h}\cdot\text{ft}\cdot{}^\circ\text{F})\frac{(520 - 530.9)\ \text{R}}{0.5\ \text{ft}} = \mathbf{-26.2\ Btu/h\cdot ft^2} \quad \text{(per unit area)}$$

Discussion The negative sign indicates that the direction of heat transfer is from the outside to the inside. Therefore, the structure is gaining heat.

2-124E A large plane wall is subjected to a specified temperature on the left (inner) surface and heat loss by radiation to space on the right (outer) surface. The temperature of the right surface of the wall and the rate of heat transfer are to be determined when steady operating conditions are reached.

Assumptions **1** Steady operating conditions are reached. **2** Heat transfer is one-dimensional since the wall is large relative to its thickness, and the thermal conditions on both sides of the wall are uniform. **3** Thermal properties are constant. **4** There is no heat generation in the wall.



Properties The properties of the plate are given to be k = 1.2 Btu/h·ft·°F and ε = 0.80.

Analysis In steady operation, heat conduction through the wall must be equal to net heat transfer from the outer surface. Therefore, taking the outer surface temperature of the plate to be T_2 (absolute, in *R*),

$$kA_s \frac{T_1 - T_2}{L} = \varepsilon\sigma A_s T_2^4$$

Canceling the area *A* and substituting the known quantities,

$$(1.2 \text{ Btu/h}\cdot\text{ft}\cdot{}^\circ\text{F}) \frac{(520 \text{ R}) - T_2}{0.5 \text{ ft}} = 0.8(0.1714 \times 10^{-8} \text{ Btu/h}\cdot\text{ft}^2\cdot\text{R}^4) T_2^4$$

Solving for T_2 gives the outer surface temperature to be T_2 = **487.7 R**

Then the rate of heat transfer through the wall becomes

$$\dot{q} = k\frac{T_1 - T_2}{L} = (1.2 \text{ Btu/h}\cdot\text{ft}\cdot{}^\circ\text{F}) \frac{(520 - 487.7) \text{ R}}{0.5 \text{ ft}} = \mathbf{77.5\ Btu/h\cdot ft^2} \quad \text{(per unit area)}$$

Discussion The positive sign indicates that the direction of heat transfer is from the inside to the outside. Therefore, the structure is losing heat as expected.

2-125 A steam pipe is subjected to convection on both the inner and outer surfaces. The mathematical formulation of the problem and expressions for the variation of temperature in the pipe and on the outer surface temperature are to be obtained for steady one-dimensional heat transfer.

Assumptions **1** Heat conduction is steady and one-dimensional since the pipe is long relative to its thickness, and there is thermal symmetry about the center line. **2** Thermal conductivity is constant. **3** There is no heat generation in the pipe.

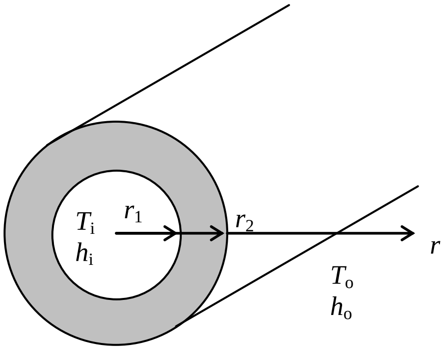
Analysis (*a*) Noting that heat transfer is steady and one-dimensional in the radial *r* direction, the mathematical formulation of this problem can be expressed as

$$\frac{d}{dr}\left(r\frac{dT}{dr}\right)=0$$

and

$$-k\frac{dT(r_1)}{dr}=h_i[T_i-T(r_1)]$$

$$-k\frac{dT(r_2)}{dr}=h_o[T(r_2)-T_o]$$



(*b*) Integrating the differential equation once with respect to *r* gives

$$r\frac{dT}{dr}=C_1$$

Dividing both sides of the equation above by *r* to bring it to a readily integrable form and then integrating,

$$\frac{dT}{dr}=\frac{C_1}{r}$$

$$T(r)=C_1\ln r+C_2$$

where C_1 and C_2 are arbitrary constants. Applying the boundary conditions give

$r = r_1$: $$-k\frac{C_1}{r_1}=h_i[T_i-(C_1\ln r_1+C_2)]$$

$r = r_2$: $$-k\frac{C_1}{r_2}=h_o[(C_1\ln r_2+C_2)-T_o]$$

Solving for C_1 and C_2 simultaneously gives

$$C_1=\frac{T_0-T_i}{\ln\frac{r_2}{r_1}+\frac{k}{h_ir_1}+\frac{k}{h_or_2}} \quad \text{and} \quad C_2=T_i-C_1\left(\ln r_1-\frac{k}{h_ir_1}\right)=T_i-\frac{T_0-T_i}{\ln\frac{r_2}{r_1}+\frac{k}{h_ir_1}+\frac{k}{h_or_2}}\left(\ln r_1-\frac{k}{h_ir_1}\right)$$

Substituting C_1 and C_2 into the general solution and simplifying, we get the variation of temperature to be

$$T(r)=C_1\ln r+T_i-C_1(\ln r_1-\frac{k}{h_ir_1})=T_i+\frac{\ln\frac{r}{r_1}+\frac{k}{h_ir_1}}{\ln\frac{r_2}{r_1}+\frac{k}{h_ir_1}+\frac{k}{h_or_2}}$$

(*c*) The outer surface temperature is determined by simply replacing *r* in the relation above by r_2. We get

$$T(r_2)=T_i+\frac{\ln\frac{r_2}{r_1}+\frac{k}{h_ir_1}}{\ln\frac{r_2}{r_1}+\frac{k}{h_ir_1}+\frac{k}{h_or_2}}$$