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سایت آموزش مهندسی مکانیک

Chapter 5

NUMERICAL METHODS IN HEAT CONDUCTION

Why Numerical Methods

5-1C Analytical solution methods are limited to *highly simplified problems* in *simple geometries*. The geometry must be such that its entire surface can be described mathematically in a coordinate system by setting the variables equal to constants. Also, heat transfer problems can not be solved analytically if the *thermal conditions* are not sufficiently simple. For example, the consideration of the variation of thermal conductivity with temperature, the variation of the heat transfer coefficient over the surface, or the radiation heat transfer on the surfaces can make it impossible to obtain an analytical solution. Therefore, analytical solutions are limited to problems that are simple or can be simplified with reasonable approximations.

5-2C The *analytical solutions* are based on (1) driving the governing differential equation by performing an energy balance on a differential volume element, (2) expressing the boundary conditions in the proper mathematical form, and (3) solving the differential equation and applying the boundary conditions to determine the integration constants. The *numerical solution* methods are based on replacing the *differential equations* by *algebraic equations*. In the case of the popular *finite difference* method, this is done by replacing the *derivatives* by *differences*. The analytical methods are simple and they provide solution functions applicable to the entire medium, but they are limited to simple problems in simple geometries. The numerical methods are usually more involved and the solutions are obtained at a number of points, but they are applicable to any geometry subjected to any kind of thermal conditions.

5-3C The *energy balance method* is based on *subdividing* the medium into a sufficient number of volume elements, and then applying an *energy balance* on each element. The formal *finite difference method* is based on replacing derivatives by their finite difference approximations. For a specified nodal network, these two methods will result in the same set of equations.

5-4C In practice, we are most likely to use a software package to solve heat transfer problems even when analytical solutions are available since we can do parametric studies very easily and present the results graphically by the press of a button. Besides, once a person is used to solving problems numerically, it is very difficult to go back to solving differential equations by hand.

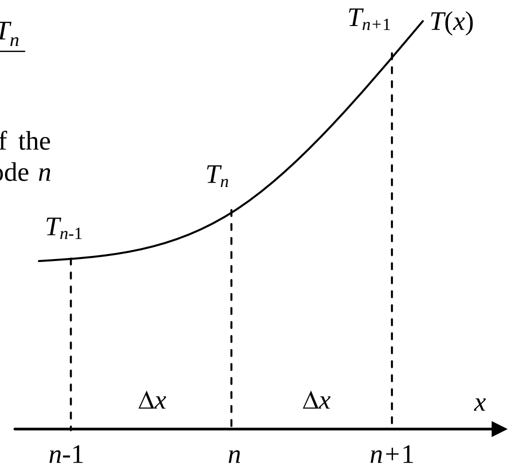
5-5C The experiments will most likely prove engineer B right since an approximate solution of a more realistic model is more accurate than the exact solution of a crude model of an actual problem.

Finite Difference Formulation of Differential Equations

5-6C A point at which the finite difference formulation of a problem is obtained is called a *node*, and all the nodes for a problem constitute the *nodal network*. The region about a node whose properties are represented by the property values at the nodal point is called the *volume element*. The distance between two consecutive nodes is called the *nodal spacing*, and a differential equation whose derivatives are replaced by differences is called a *difference equation*.

5-7 We consider three consecutive nodes n-1, n, and n+1 in a plain wall. Using Eq. 5-6, the first derivative of temperature dT/dx at the midpoints n - 1/2 and n + 1/2 of the sections surrounding the node n can be expressed as

$$\left.\frac{dT}{dx}\right|_{n-\frac{1}{2}} \cong \frac{T_n - T_{n-1}}{\Delta x} \quad \text{and} \quad \left.\frac{dT}{dx}\right|_{n+\frac{1}{2}} \cong \frac{T_{n+1} - T_n}{\Delta x}$$



Noting that second derivative is simply the derivative of the first derivative, the second derivative of temperature at node n can be expressed as

$$\left.\frac{d^2T}{dx^2}\right|_n \cong \frac{\left.\frac{dT}{dx}\right|_{n+\frac{1}{2}} - \left.\frac{dT}{dx}\right|_{n-\frac{1}{2}}}{\Delta x}$$

$$= \frac{\frac{T_{n+1} - T_n}{\Delta x} - \frac{T_n - T_{n-1}}{\Delta x}}{\Delta x} = \frac{T_{n-1} - 2T_n + T_{n+1}}{\Delta x^2}$$

which is the *finite difference representation* of the *second derivative* at a general internal node n. Note that the second derivative of temperature at a node n is expressed in terms of the temperatures at node n and its two neighboring nodes

5-8 The finite difference formulation of steady two-dimensional heat conduction in a medium with heat generation and constant thermal conductivity is given by

$$\frac{T_{m-1,n} - 2T_{m,n} + T_{m+1,n}}{\Delta x^2} + \frac{T_{m,n-1} - 2T_{m,n} + T_{m,n+1}}{\Delta y^2} + \frac{\dot{g}_{m,n}}{k} = 0$$

in rectangular coordinates. This relation can be modified for the three-dimensional case by simply adding another index j to the temperature in the z direction, and another difference term for the z direction as

$$\frac{T_{m-1,n,j} - 2T_{m,n,j} + T_{m+1,n,j}}{\Delta x^2} + \frac{T_{m,n-1,j} - 2T_{m,n,j} + T_{m,n+1,j}}{\Delta y^2} + \frac{T_{m,n,j-1} - 2T_{m,n,j} + T_{m,n,j+1}}{\Delta z^2} + \frac{\dot{g}_{m,n,j}}{k} = 0$$

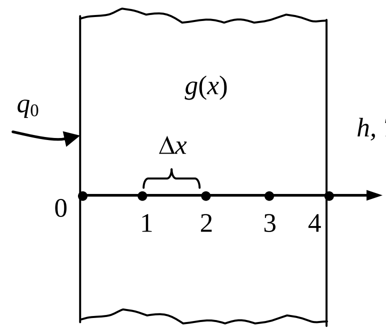
5-9 A plane wall with variable heat generation and constant thermal conductivity is subjected to uniform heat flux $\dot{q}_0$ at the left (node 0) and convection at the right boundary (node 4). Using the finite difference form of the 1st derivative, the finite difference formulation of the boundary nodes is to be determined.

Assumptions **1** Heat transfer through the wall is steady since there is no indication of change with time. **2** Heat transfer is one-dimensional since the plate is large relative to its thickness. **3** Thermal conductivity is constant and there is nonuniform heat generation in the medium. **4** Radiation heat transfer is negligible.

Analysis The boundary conditions at the left and right boundaries can be expressed analytically as

At x = 0: $$-k\frac{dT(0)}{dx} = q_0$$

At $x = L$: $$-k\frac{dT(L)}{dx} = h[T(L) - T_\infty]$$



Replacing derivatives by differences using values at the closest nodes, the finite difference form of the 1st derivative of temperature at the boundaries (nodes 0 and 4) can be expressed as

$$\left.\frac{dT}{dx}\right|_{\text{left, m}=0} \cong \frac{T_1 - T_0}{\Delta x} \quad \text{and} \quad \left.\frac{dT}{dx}\right|_{\text{right, m}=4} \cong \frac{T_4 - T_3}{\Delta x}$$

Substituting, the finite difference formulation of the boundary nodes become

At x = 0: $$-k\frac{T_1 - T_0}{\Delta x} = q_0$$

At $x = L$: $$-k\frac{T_4 - T_3}{\Delta x} = h[T_4 - T_\infty]$$

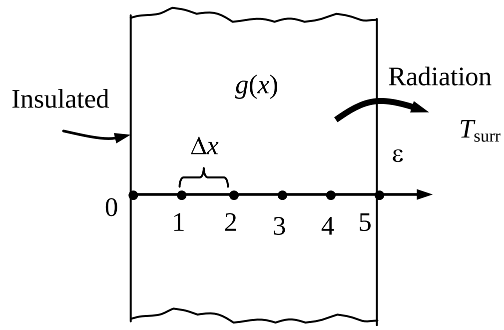
5-10 A plane wall with variable heat generation and constant thermal conductivity is subjected to insulation at the left (node 0) and radiation at the right boundary (node 5). Using the finite difference form of the 1st derivative, the finite difference formulation of the boundary nodes is to be determined.

Assumptions **1** Heat transfer through the wall is steady since there is no indication of change with time. **2** Heat transfer is one-dimensional since the plate is large relative to its thickness. **3** Thermal conductivity is constant and there is nonuniform heat generation in the medium. **4** Convection heat transfer is negligible.

Analysis The boundary conditions at the left and right boundaries can be expressed analytically as

At $x = 0$: $$-k\frac{dT(0)}{dx} = 0 \quad \text{or} \quad \frac{dT(0)}{dx} = 0$$

At $x = L$: $$-k\frac{dT(L)}{dx} = \varepsilon\sigma[T^4(L) - T_{surr}^4]$$



Replacing derivatives by differences using values at the closest nodes, the finite difference form of the 1st derivative of temperature at the boundaries (nodes 0 and 5) can be expressed as

$$\left.\frac{dT}{dx}\right|_{\text{left, m}=0} \cong \frac{T_1 - T_0}{\Delta x} \quad \text{and} \quad \left.\frac{dT}{dx}\right|_{\text{right, m}=5} \cong \frac{T_5 - T_4}{\Delta x}$$

Substituting, the finite difference formulation of the boundary nodes become

At $x = 0$: $$-k\frac{T_1 - T_0}{\Delta x} = 0 \qquad \text{or} \qquad T_1 = T_0$$

At $x = L$: $$-k\frac{T_5 - T_4}{\Delta x} = \varepsilon\sigma[T_5^4 - T_{surr}^4]$$

One-Dimensional Steady Heat Conduction

5-11C The finite difference form of a heat conduction problem by the *energy balance method* is obtained by *subdividing* the medium into a sufficient number of volume elements, and then applying an *energy balance* on each element. This is done by first *selecting* the nodal points (or nodes) at which the temperatures are to be determined, and then *forming elements* (or control volumes) over the nodes by drawing lines through the midpoints between the nodes. The properties *at the node* such as the temperature and the rate of heat generation represent the *average* properties of the element. The temperature is assumed to vary *linearly* between the nodes, especially when expressing heat conduction between the elements using Fourier's law.

5-12C In the energy balance formulation of the finite difference method, it is recommended that all heat transfer at the boundaries of the volume element be assumed to be *into* the volume element even for steady heat conduction. This is a valid recommendation even though it seems to violate the conservation of energy principle since the assumed direction of heat conduction at the surfaces of the volume elements has no effect on the formulation, and some heat conduction terms turn out to be negative.

5-13C In the finite difference formulation of a problem, an insulated boundary is best handled by replacing the insulation by a mirror, and treating the node on the boundary as an *interior* node. Also, a thermal symmetry line and an insulated boundary are treated the same way in the finite difference formulation.

5-14C A node on an insulated boundary can be treated as an interior node in the finite difference formulation of a plane wall by replacing the insulation on the boundary by a *mirror*, and considering the reflection of the medium as its extension. This way the node next to the boundary node appears on both sides of the boundary node because of symmetry, converting it into an interior node.

5-15C In a medium in which the finite difference formulation of a general interior node is given in its simplest form as

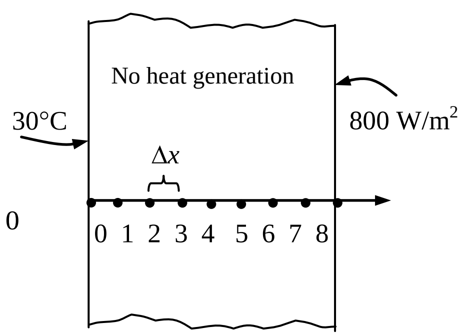
$$\frac{T_{m-1} - 2T_m + T_{m+1}}{\Delta x^2} + \frac{\dot{g}_m}{k} = 0$$

(*a*) heat transfer in this medium is **steady**, (*b*) it is **one-dimensional**, (*c*) there **is** heat generation, (*d*) the nodal spacing is **constant**, and (*e*) the thermal conductivity is **constant**.

5-16 A plane wall with no heat generation is subjected to specified temperature at the left (node 0) and heat flux at the right boundary (node 8). The finite difference formulation of the boundary nodes and the finite difference formulation for the rate of heat transfer at the left boundary are to be determined.

Assumptions **1** Heat transfer through the wall is given to be steady, and the thermal conductivity to be constant. **2** Heat transfer is one-dimensional since the plate is large relative to its thickness. **3** There is no heat generation in the medium.

Analysis Using the energy balance approach and taking the direction of all heat transfers to be towards the node under consideration, the finite difference formulations become



Left boundary node: $T_0 = 30$

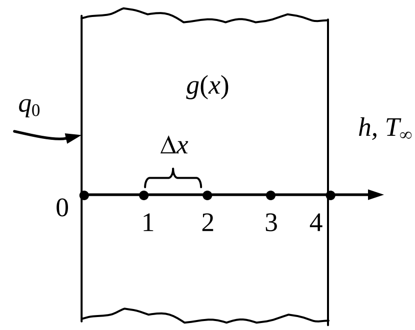
Right boundary node: $kA\frac{T_7 - T_8}{\Delta x} + \dot{q}_0 A = 0$ or $k\frac{T_7 - T_8}{\Delta x} + 800 = 0$

Heat transfer at left surface: $\dot{Q}_{\text{left surface}} + kA\frac{T_1 - T_0}{\Delta x} = 0$

5-17 A plane wall with variable heat generation and constant thermal conductivity is subjected to uniform heat flux $\dot{q}_0$ at the left (node 0) and convection at the right boundary (node 4). The finite difference formulation of the boundary nodes is to be determined.

Assumptions **1** Heat transfer through the wall is given to be steady, and the thermal conductivity to be constant. **2** Heat transfer is one-dimensional since the plate is large relative to its thickness. **3** Radiation heat transfer is negligible.

Analysis Using the energy balance approach and taking the direction of all heat transfers to be towards the node under consideration, the finite difference formulations become

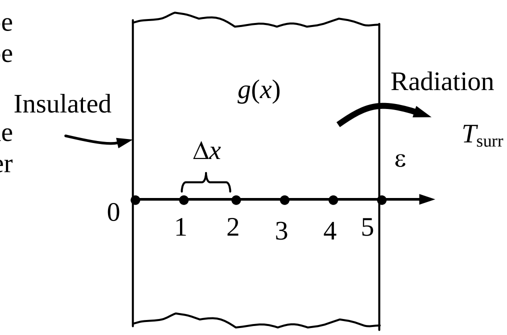


Left boundary node: $\dot{q}_0 A + kA\dfrac{T_1 - T_0}{\Delta x} + \dot{g}_0 (A\Delta x/2) = 0$

Right boundary node: $kA\dfrac{T_3 - T_4}{\Delta x} + hA(T_\infty - T_4) + \dot{g}_4 (A\Delta x/2) = 0$

5-18 A plane wall with variable heat generation and constant thermal conductivity is subjected to insulation at the left (node 0) and radiation at the right boundary (node 5). The finite difference formulation of the boundary nodes is to be determined.

Assumptions **1** Heat transfer through the wall is given to be steady and one-dimensional, and the thermal conductivity to be constant. **2** Convection heat transfer is negligible.

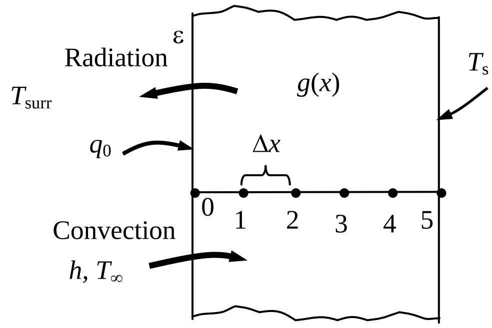


Analysis Using the energy balance approach and taking the direction of all heat transfers to be towards the node under consideration, the finite difference formulations become

Left boundary node: $kA\dfrac{T_1 - T_0}{\Delta x} + \dot{g}_0 (A\Delta x/2) = 0$

Right boundary node: $\varepsilon\sigma A(T_{\rm surr}^4 - T_5^4) + kA\dfrac{T_4 - T_5}{\Delta x} + \dot{g}_5 (A\Delta x/2) = 0$

5-19 A plane wall with variable heat generation and constant thermal conductivity is subjected to combined convection, radiation, and heat flux at the left (node 0) and specified temperature at the right boundary (node 5). The finite difference formulation of the left boundary node (node 0) and the finite difference formulation for the rate of heat transfer at the right boundary (node 5) are to be determined.



Assumptions **1** Heat transfer through the wall is given to be steady and one-dimensional. **2** The thermal conductivity is given to be constant.

Analysis Using the energy balance approach and taking the direction of all heat transfers to be towards the node under consideration, the finite difference formulations become

Left boundary node (all temperatures are in K):

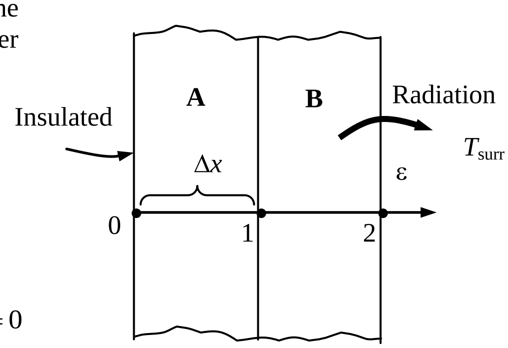
$$\varepsilon\sigma A(T_{surr}^4 - T_0^4) + hA(T_\infty - T_0) + kA\frac{T_1 - T_0}{\Delta x} + \dot{q}_0 A + \dot{g}_0 (A\Delta x/2) = 0$$

Heat transfer at right surface: $\dot{Q}_{\text{right surface}} + kA\dfrac{T_4 - T_5}{\Delta x} + \dot{g}_5 (A\Delta x/2) = 0$

5-20 A composite plane wall consists of two layers A and B in perfect contact at the interface where node 1 is. The wall is insulated at the left (node 0) and subjected to radiation at the right boundary (node 2). The complete finite difference formulation of this problem is to be obtained.

Assumptions **1** Heat transfer through the wall is given to be steady and one-dimensional, and the thermal conductivity to be constant. **2** Convection heat transfer is negligible. **3** There is no heat generation.

Analysis Using the energy balance approach and taking the direction of all heat transfers to be towards the node under consideration, the finite difference formulations become

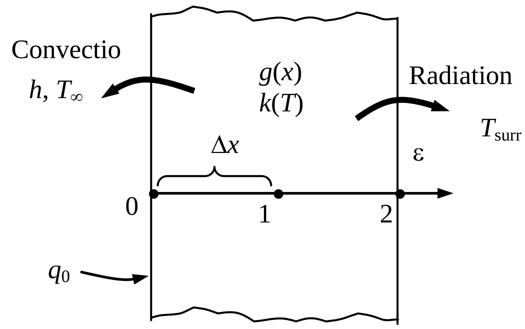


Node 0 (at left boundary): $k_A A\dfrac{T_1 - T_0}{\Delta x} = 0 \;\rightarrow\; T_1 = T_0$

Node 1 (at the interface): $k_A A\dfrac{T_0 - T_1}{\Delta x} + k_B A\dfrac{T_2 - T_1}{\Delta x} = 0$

Node 2 (at right boundary): $\varepsilon\sigma A(T_{surr}^4 - T_2^4) + k_B A\dfrac{T_1 - T_2}{\Delta x} = 0$

5-21 A plane wall with variable heat generation and variable thermal conductivity is subjected to specified heat flux $\dot{q}_0$ and convection at the left boundary (node 0) and radiation at the right boundary (node 5). The complete finite difference formulation of this problem is to be obtained.



Assumptions **1** Heat transfer through the wall is given to be steady and one-dimensional, and the thermal conductivity and heat generation to be variable. **2** Convection heat transfer at the right surface is negligible.

Analysis Using the energy balance approach and taking the direction of all heat transfers to be towards the node under consideration, the finite difference formulations become

Node 0 (at left boundary): $\dot{q}_0 A + hA(T_\infty - T_0) + k_0 A\dfrac{T_1 - T_0}{\Delta x} + \dot{g}_0(A\Delta x/2) = 0$

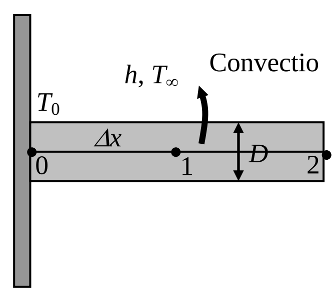
Node 1 (at the mid plane): $k_1 A\dfrac{T_0 - T_1}{\Delta x} + k_1 A\dfrac{T_2 - T_1}{\Delta x} + \dot{g}_1(A\Delta x/2) = 0$

Node 2 (at right boundary): $\varepsilon\sigma A(T_{surr}^4 - T_2^4) + k_2 A\dfrac{T_1 - T_2}{\Delta x} + \dot{g}_2(A\Delta x/2) = 0$

5-22 A pin fin with negligible heat transfer from its tip is considered. The complete finite difference formulation for the determination of nodal temperatures is to be obtained.

Assumptions **1** Heat transfer through the pin fin is given to be steady and one-dimensional, and the thermal conductivity to be constant. **2** Convection heat transfer coefficient is constant and uniform. **3** Radiation heat transfer is negligible. **4** Heat loss from the fin tip is given to be negligible.

Analysis The nodal network consists of 3 nodes, and the base temperature T_0 at node 0 is specified. Therefore, there are two unknowns T_1 and T_2, and we need two equations to determine them. Using the energy balance approach and taking the direction of all heat transfers to be towards the node under consideration, the finite difference formulations become



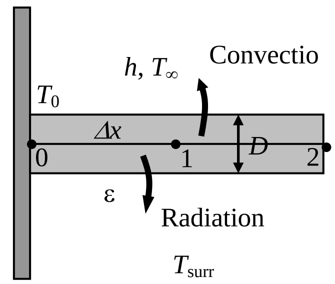
Node 1 (at midpoint): $kA\dfrac{T_0 - T_1}{\Delta x} + kA\dfrac{T_2 - T_1}{\Delta x} + hp\Delta x(T_\infty - T_1) = 0$

Node 2 (at fin tip): $kA\dfrac{T_1 - T_2}{\Delta x} + h(p\Delta x/2)(T_\infty - T_2) = 0$

where $A = \pi D^2/4$ is the cross-sectional area and $p = \pi D$ is the perimeter of the fin.

5-23 A pin fin with negligible heat transfer from its tip is considered. The complete finite difference formulation for the determination of nodal temperatures is to be obtained.

Assumptions **1** Heat transfer through the pin fin is given to be steady and one-dimensional, and the thermal conductivity to be constant. **2** Convection heat transfer coefficient is constant and uniform. **3** Heat loss from the fin tip is given to benegligible.



Analysis The nodal network consists of 3 nodes, and the base temperature T_0 at node 0 is specified. Therefore, there are two unknowns T_1 and T_2, and we need two equations to determine them. Using the energy balance approach and taking the direction of all heat transfers to be towards the node under consideration, the finite difference formulations become

Node 1 (at midpoint): $$kA\frac{T_0-T_1}{\Delta x}+kA\frac{T_2-T_1}{\Delta x}+h(p\Delta x/2)(T_\infty-T_1)+\varepsilon\sigma A(T_{surr}^4-T_1^4)=0$$

Node 2 (at fin tip): $$kA\frac{T_1-T_2}{\Delta x}+h(p\Delta x/2)(T_\infty-T_2)+\varepsilon\sigma(p\Delta x/2)(T_{surr}^4-T_2^4)=0$$

where $A=\pi D^2/4$ is the cross-sectional area and $p=\pi D$ is the perimeter of the fin.

5-24 A uranium plate is subjected to insulation on one side and convection on the other. The finite difference formulation of this problem is to be obtained, and the nodal temperatures under steady conditions are to be determined.

Assumptions **1** Heat transfer through the wall is steady since there is no indication of change with time. **2** Heat transfer is one-dimensional since the plate is large relative to its thickness. **3** Thermal conductivity is constant. **4** Radiation heat transfer is negligible.

Properties The thermal conductivity is given to be k = 28 W/m·°C.

Analysis The number of nodes is specified to be M = 6. Then the nodal spacing Δx becomes

$$\Delta x = \frac{L}{M-1} = \frac{0.05\,\text{m}}{6\text{-}1} = 0.01\,\text{m}$$

This problem involves 6 unknown nodal temperatures, and thus we need to have 6 equations to determine them uniquely. Node 0 is on insulated boundary, and thus we can treat it as an interior note by using the mirror image concept. Nodes 1, 2, 3, and 4 are interior nodes, and thus for them we can use the general finite difference relation expressed as

$$\frac{T_{m-1} - 2T_m + T_{m+1}}{\Delta x^2} + \frac{\dot{g}_m}{k} = 0, \quad \text{for } m = 0, 1, 2, 3, \text{and } 4$$

Finally, the finite difference equation for node 5 on the right surface subjected to convection is obtained by applying an energy balance on the half volume element about node 5 and taking the direction of all heat transfers to be towards the node under consideration:

Node 0 (Left surface - insulated): $\dfrac{T_1 - 2T_0 + T_1}{\Delta x^2} + \dfrac{\dot{g}}{k} = 0$

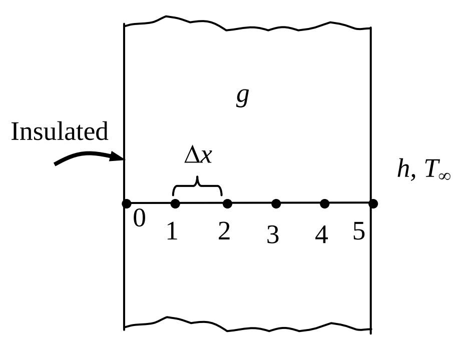
Node 1 (interior): $\dfrac{T_0 - 2T_1 + T_2}{\Delta x^2} + \dfrac{\dot{g}}{k} = 0$

Node 2 (interior): $\dfrac{T_1 - 2T_2 + T_3}{\Delta x^2} + \dfrac{\dot{g}}{k} = 0$

Node 3 (interior): $\dfrac{T_2 - 2T_3 + T_4}{\Delta x^2} + \dfrac{\dot{g}}{k} = 0$

Node 4 (interior): $\dfrac{T_3 - 2T_4 + T_5}{\Delta x^2} + \dfrac{\dot{g}}{k} = 0$

Node 5 (right surface - convection): $h(T_\infty - T_5) + k\dfrac{T_4 - T_5}{\Delta x} + \dot{g}(\Delta x/2) = 0$



where $\Delta x = 0.01\,\text{m}$, $\dot{g} = 6\times10^5$ W/m^3, $k = 28$ W/m·°C, $h = 60$ W/m^2·°C, and $T_\infty = 30$°C. This system of 6 equations with six unknown temperatures constitute the finite difference formulation of the problem.

(*b*) The 6 nodal temperatures under steady conditions are determined by solving the 6 equations above simultaneously with an equation solver to be

T_0 =556.8°C, T_1 =555.7°C, T_2 =552.5°C, T_3 =547.1°C, T_4 =539.6°C, and T_5 =530.0°C

Discussion This problem can be solved analytically by solving the differential equation as described in Chap. 2, and the analytical (exact) solution can be used to check the accuracy of the numerical solution above.

5-25 A long triangular fin attached to a surface is considered. The nodal temperatures, the rate of heat transfer, and the fin efficiency are to be determined numerically using 6 equally spaced nodes.

Assumptions **1** Heat transfer along the fin is given to be steady, and the temperature along the fin to vary in the *x* direction only so that $T = T(x)$. **2** Thermal conductivity is constant.

Properties The thermal conductivity is given to be k = 180 W/m·°C. The emissivity of the fin surface is 0.9.

Analysis The fin length is given to be L = 5 cm, and the number of nodes is specified to be M = 6. Therefore, the nodal spacing Δx is

$$\Delta x = \frac{L}{M-1} = \frac{0.05\,\text{m}}{6\text{-}1} = 0.01\,\text{m}$$

The temperature at node 0 is given to be T_0 = 200°C, and the temperatures at the remaining 5 nodes are to be determined. Therefore, we need to have 5 equations to determine them uniquely. Nodes 1, 2, 3, and 4 are interior nodes, and the finite difference formulation for a *general interior node m* is obtained by applying an energy balance on the volume element of this node. Noting that heat transfer is steady and there is no heat generation in the fin and assuming heat transfer to be into the medium from all sides, the energy balance can be expressed as

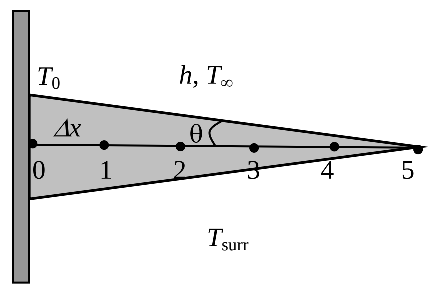
$$\sum_{\text{all sides}} \dot{Q} = 0 \quad \rightarrow \quad kA_{\text{left}} \frac{T_{m-1} - T_m}{\Delta x} + kA_{\text{right}} \frac{T_{m+1} - T_m}{\Delta x} + hA_{\text{conv}}(T_\infty - T_m) + \varepsilon\sigma A_{\text{surface}}[T_{\text{surr}}^4 - (T_m + 273)^4\} = 0$$

Note that heat transfer areas are different for each node in this case, and using geometrical relations, they can be expressed as

$$A_{\text{left}} = (\text{Height} \times \text{width})_{@\,m-1/2} = 2w\left[L - (m - 1/2)\Delta x\right]\tan\theta$$

$$A_{\text{right}} = (\text{Height} \times \text{width})_{@\,m+1/2} = 2w\left[L - (m + 1/2)\Delta x\right]\tan\theta$$

$$A_{\text{surface}} = 2 \times \text{Length} \times \text{width} = 2w(\Delta x / \cos\theta)$$



Substituting,

$$2kw[L - (m - 0.5)\Delta x]\tan\theta \frac{T_{m-1} - T_m}{\Delta x} + 2kw[L - (m + 0.5)\Delta x]\tan\theta \frac{T_{m+1} - T_m}{\Delta x}$$
$$+ 2w(\Delta x / \cos\theta)\{h(T_\infty - T_m) + \varepsilon\sigma[T_{\text{surr}}^4 - (T_m + 273)^4]\} = 0$$

Dividing each term by $2kwL\tan\theta/\Delta x$ gives

$$\left[1 - (m - 1/2)\frac{\Delta x}{L}\right](T_{m-1} - T_m) + \left[1 - (m + 1/2)\frac{\Delta x}{L}\right](T_{m+1} - T_m) + \frac{h(\Delta x)^2}{kL\sin\theta}(T_\infty - T_m) + \frac{\varepsilon\sigma(\Delta x)^2}{kL\sin\theta}[T_{\text{surr}}^4 - (T_m + 273)^4] = 0$$

Substituting,

m = 1: $$\left[1 - 0.5\frac{\Delta x}{L}\right](T_0 - T_1) + \left[1 - 1.5\frac{\Delta x}{L}\right](T_2 - T_1) + \frac{h(\Delta x)^2}{kL\sin\theta}(T_\infty - T_1) + \frac{\varepsilon\sigma(\Delta x)^2}{kL\sin\theta}[T_{\text{surr}}^4 - (T_1 + 273)^4] = 0$$

m = 2: $$\left[1 - 1.5\frac{\Delta x}{L}\right](T_1 - T_2) + \left[1 - 2.5\frac{\Delta x}{L}\right](T_3 - T_2) + \frac{h(\Delta x)^2}{kL\sin\theta}(T_\infty - T_2) + \frac{\varepsilon\sigma(\Delta x)^2}{kL\sin\theta}[T_{\text{surr}}^4 - (T_2 + 273)^4] = 0$$

m = 3: $$\left[1 - 2.5\frac{\Delta x}{L}\right](T_2 - T_3) + \left[1 - 3.5\frac{\Delta x}{L}\right](T_4 - T_3) + \frac{h(\Delta x)^2}{kL\sin\theta}(T_\infty - T_3) + \frac{\varepsilon\sigma(\Delta x)^2}{kL\sin\theta}[T_{\text{surr}}^4 - (T_3 + 273)^4] = 0$$

m = 4: $$\left[1 - 3.5\frac{\Delta x}{L}\right](T_3 - T_4) + \left[1 - 4.5\frac{\Delta x}{L}\right](T_5 - T_4) + \frac{h(\Delta x)^2}{kL\sin\theta}(T_\infty - T_4) + \frac{\varepsilon\sigma(\Delta x)^2}{kL\sin\theta}[T_{\text{surr}}^4 - (T_4 + 273)^4] = 0$$

An energy balance on the 5th node gives the 5th equation,

m = 5: $$2k\frac{\Delta x}{2}\tan\theta \frac{T_4 - T_5}{\Delta x} + 2h\frac{\Delta x / 2}{\cos\theta}(T_\infty - T_5) + 2\varepsilon\sigma\frac{\Delta x / 2}{\cos\theta}[T_{\text{surr}}^4 - (T_5 + 273)^4] = 0$$

Solving the 5 equations above simultaneously for the 5 unknown nodal temperatures gives

$$\boldsymbol{T_1 = 177.0°\text{C}, \quad T_2 = 174.1°\text{C}, \quad T_3 = 171.2°\text{C}, \quad T_4 = 168.4°\text{C}, \quad \text{and} \quad T_5 = 165.5°\text{C}}$$

(*b*) The total rate of heat transfer from the fin is simply the sum of the heat transfer from each volume element to the ambient, and for w = 1 m it is determined from

$$\dot{Q}_{\text{fin}} = \sum_{m=0}^{5} \dot{Q}_{\text{element},m} = \sum_{m=0}^{5} hA_{\text{surface},m}(T_m - T_\infty) + \sum_{m=0}^{5} \varepsilon\sigma A_{\text{surface},m}[(T_m + 273)^4 - T_{\text{surr}}^4]$$

Noting that the heat transfer surface area is $w\Delta x/\cos\theta$ for the boundary nodes 0 and 5, and twice as large for the interior nodes 1, 2, 3, and 4, we have

$$\begin{aligned}
\dot{Q}_{\text{fin}} &= h\frac{w\Delta x}{\cos\theta}\left[(T_0 - T_\infty) + 2(T_1 - T_\infty) + 2(T_2 - T_\infty) + 2(T_3 - T_\infty) + 2(T_4 - T_\infty) + (T_5 - T_\infty)\right] \\
&\quad + \varepsilon\sigma\frac{w\Delta x}{\cos\theta}\{[(T_0 + 273)^4 - T_{\text{surr}}^4] + 2[(T_1 + 273)^4 - T_{\text{surr}}^4] + 2[(T_2 + 273)^4 - T_{\text{surr}}^4] + 2[(T_3 + 273)^4 - T_{\text{surr}}^4] \\
&\quad + 2[(T_4 + 273)^4 - T_{\text{surr}}^4] + [(T_5 + 273)^4 - T_{\text{surr}}^4]\} \\
&= \mathbf{533\ W}
\end{aligned}$$

5-26 "!PROBLEM 5-26"

```
"GIVEN"
k=180 "[W/m-C]"
L=0.05 "[m]"
b=0.01 "[m]"
w=1 "[m]"
"T_0=180 [C], parameter to be varied"
T_infinity=25 "[C]"
h=25 "[W/m^2-C]"
T_surr=290 "[K]"
M=6
epsilon=0.9
tan(theta)=(0.5*b)/L
sigma=5.67E-8 "[W/m^2-K^4], Stefan-Boltzmann constant"

"ANALYSIS"
"(a)"
DELTAx=L/(M-1)
"Using the finite difference method, the five equations for the temperatures at 5 nodes are
determined to be"
(1-0.5*DELTAx/L)*(T_0-T_1)+(1-1.5*DELTAx/L)*(T_2-
T_1)+(h*DELTAx^2)/(k*L*sin(theta))*(T_infinity-
T_1)+(epsilon*sigma*DELTAX^2)/(k*L*sin(theta))*(T_surr^4-(T_1+273)^4)=0 "for mode 1"

(1-1.5*DELTAx/L)*(T_1-T_2)+(1-2.5*DELTAx/L)*(T_3-
T_2)+(h*DELTAx^2)/(k*L*sin(theta))*(T_infinity-
T_2)+(epsilon*sigma*DELTAX^2)/(k*L*sin(theta))*(T_surr^4-(T_2+273)^4)=0 "for mode 2"

(1-2.5*DELTAx/L)*(T_2-T_3)+(1-3.5*DELTAx/L)*(T_4-
T_3)+(h*DELTAx^2)/(k*L*sin(theta))*(T_infinity-
T_3)+(epsilon*sigma*DELTAX^2)/(k*L*sin(theta))*(T_surr^4-(T_3+273)^4)=0 "for mode 3"

(1-3.5*DELTAx/L)*(T_3-T_4)+(1-4.5*DELTAx/L)*(T_5-
T_4)+(h*DELTAx^2)/(k*L*sin(theta))*(T_infinity-
T_4)+(epsilon*sigma*DELTAX^2)/(k*L*sin(theta))*(T_surr^4-(T_4+273)^4)=0 "for mode 4"

2*k*DELTAx/2*tan(theta)*(T_4-T_5)/DELTAx+2*h*(0.5*DELTAx)/cos(theta)*(T_infinity-
T_5)+2*epsilon*sigma*(0.5*DELTAx)/cos(theta)*(T_surr^4-(T_5+273)^4)=0 "for mode 5"

T_tip=T_5

"(b)"
Q_dot_fin=C+D "where"

C=h*(w*DELTAx)/cos(theta)*((T_0-T_infinity)+2*(T_1-T_infinity)+2*(T_2-T_infinity)+2*(T_3-
T_infinity)+2*(T_4-T_infinity)+(T_5-T_infinity))

D=epsilon*sigma*(w*DELTAx)/cos(theta)*(((T_0+273)^4-T_surr^4)+2*((T_1+273)^4-
T_surr^4)+2*((T_2+273)^4-T_surr^4)+2*((T_3+273)^4-T_surr^4)+2*((T_4+273)^4-
T_surr^4)+((T_5+273)^4-T_surr^4))
```

T_0 [C]	**T_{tip} [C]**	**Q_{fin} [W]**
100	93.51	239.8
105	98.05	256.8
110	102.6	274
115	107.1	291.4
120	111.6	309
125	116.2	326.8
130	120.7	344.8
135	125.2	363.1
140	129.7	381.5
145	134.2	400.1
150	138.7	419
155	143.2	438.1
160	147.7	457.5
165	152.1	477.1
170	156.6	496.9
175	161.1	517
180	165.5	537.3
185	170	557.9
190	174.4	578.7
195	178.9	599.9
200	183.3	621.2

5-27 A plate is subjected to specified temperature on one side and convection on the other. The finite difference formulation of this problem is to be obtained, and the nodal temperatures under steady conditions as well as the rate of heat transfer through the wall are to be determined.

Assumptions **1** Heat transfer through the wall is given to be steady and one-dimensional. **2** Thermal conductivity is constant. **3** There is no heat generation. **4** Radiation heat transfer is negligible.

Properties The thermal conductivity is given to be k = 2.3 W/m·°C.

Analysis The nodal spacing is given to be Δx=0.1 m. Then the number of nodes M becomes

$$M = \frac{L}{\Delta x} + 1 = \frac{0.4\text{ m}}{0.1\text{ m}} + 1 = 5$$

The left surface temperature is given to be T_0 =80°C. This problem involves 4 unknown nodal temperatures, and thus we need to have 4 equations to determine them uniquely. Nodes 1, 2, and 3 are interior nodes, and thus for them we can use the general finite difference relation expressed as

$$\frac{T_{m-1} - 2T_m + T_{m+1}}{\Delta x^2} + \frac{\dot{g}_m}{k} = 0 \rightarrow T_{m-1} - 2T_m + T_{m+1} = 0 \quad (\text{since } \dot{g} = 0), \quad \text{for } m = 0, 1, 2, \text{and } 3$$

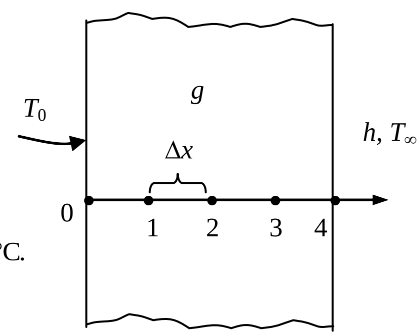
The finite difference equation for node 4 on the right surface subjected to convection is obtained by applying an energy balance on the half volume element about node 4 and taking the direction of all heat transfers to be towards the node under consideration:

Node 1 (interior): $T_0 - 2T_1 + T_2 = 0$

Node 2 (interior): $T_1 - 2T_2 + T_3 = 0$

Node 3 (interior): $T_2 - 2T_3 + T_4 = 0$

Node 4 (right surface - convection): $h(T_\infty - T_4) + k\dfrac{T_3 - T_4}{\Delta x} = 0$



where $\Delta x = 0.1\text{ m}$, $k = 2.3\text{ W/m}\cdot°\text{C}$, $h = 24\text{ W/m}^2\cdot°\text{C}$, and $T_\infty = 15°\text{C}$.

The system of 4 equations with 4 unknown temperatures constitute the finite difference formulation of the problem.

(b) The nodal temperatures under steady conditions are determined by solving the 4 equations above simultaneously with an equation solver to be

T_1 =66.9°C, T_2 =53.8°C, T_3 =40.7°C, and T_4 =27.6°C

(c) The rate of heat transfer through the wall is simply convection heat transfer at the right surface,

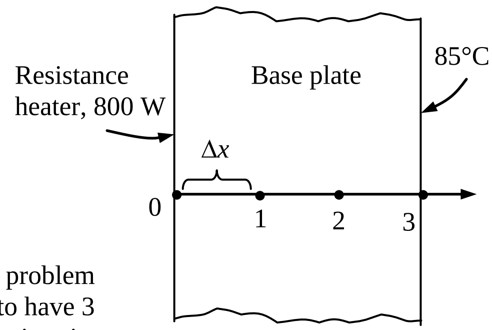
$$\dot{Q}_{\text{wall}} = \dot{Q}_{\text{conv}} = hA(T_4 - T_\infty) = (24\text{ W/m}^2.°\text{C})(20\text{ m}^2)(27.56\text{-}15)°\text{C} = \mathbf{6029\text{ W}}$$

Discussion This problem can be solved analytically by solving the differential equation as described in Chap. 2, and the analytical (exact) solution can be used to check the accuracy of the numerical solution above.

5-28 A plate is subjected to specified heat flux on one side and specified temperature on the other. The finite difference formulation of this problem is to be obtained, and the unknown surface temperature under steady conditions is to be determined.

Assumptions **1** Heat transfer through the base plate is given to be steady. **2** Heat transfer is one-dimensional since the plate is large relative to its thickness. **3** There is no heat generation in the plate. **4** Radiation heat transfer is negligible. **5** The entire heat generated by the resistance heaters is transferred through the plate.

Properties The thermal conductivity is given to be k = 20 W/m·°C.



Analysis The nodal spacing is given to be Δx=0.2 cm. Then the number of nodes M becomes

$$M = \frac{L}{\Delta x} + 1 = \frac{0.6\,\text{cm}}{0.2\,\text{cm}} + 1 = 4$$

The right surface temperature is given to be T_3 =85°C. This problem involves 3 unknown nodal temperatures, and thus we need to have 3 equations to determine them uniquely. Nodes 1, 2, and 3 are interior nodes, and thus for them we can use the general finite difference relation expressed as

$$\frac{T_{m-1} - 2T_m + T_{m+1}}{\Delta x^2} + \frac{\dot{g}_m}{k} = 0 \rightarrow T_{m-1} - 2T_m + T_{m+1} = 0 \quad (\text{since } \dot{g} = 0)\,, \quad \text{for } m = 1 \text{ and } 2$$

The finite difference equation for node 0 on the left surface subjected to uniform heat flux is obtained by applying an energy balance on the half volume element about node 0 and taking the direction of all heat transfers to be towards the node under consideration:

$$\begin{aligned}
&\text{Node 4 (right surface - convection):} && \dot{q}_0 + k\frac{T_1 - T_0}{\Delta x} = 0 \\
&\text{Node 1 (interior):} && T_0 - 2T_1 + T_2 = 0 \\
&\text{Node 2 (interior):} && T_1 - 2T_2 + T_3 = 0
\end{aligned}$$

where $\Delta x = 0.2\,\text{cm},\ k = 20\,\text{W/m}\cdot°\text{C},\ T_3 = 85°\text{C, and } \dot{q}_0 = \dot{Q}_0 / A = (800\text{W})/(0.0160\,\text{m}^2) = 50{,}000\,\text{W/m}^2$.
The system of 3 equations with 3 unknown temperatures constitute the finite difference formulation of the problem.

(*b*) The nodal temperatures under steady conditions are determined by solving the 3 equations above simultaneously with an equation solver to be

$\boldsymbol{T_0}$ **=100°C**, T_1 =95°C, and T_2 =90°C

Discussion This problem can be solved analytically by solving the differential equation as described in Chap. 2, and the analytical (exact) solution can be used to check the accuracy of the numerical solution above.