Transient Heat Conduction

5-63C The formulation of a transient heat conduction problem differs from that of a steady heat conduction problem in that the transient problem involves an *additional term* that represents the *change in the energy content* of the medium with time. This additional term $\rho A \Delta x C(T_m^{i+1} - T_m^i)/\Delta t$ represent the change in the internal energy content during Δt in the transient finite difference formulation.

5-64C The two basic methods of solution of transient problems based on finite differencing are the *explicit* and the *implicit methods*. The heat transfer terms are expressed at time step *i* in the explicit method, and at the future time step *i* + 1 in the implicit method as

Explicit method: $$\sum_{\text{All sides}} \dot{Q}^i + \dot{G}^{\text{i}}_{\text{element}} = \rho V_{\text{element}} C \frac{T_m^{i+1} - T_m^i}{\Delta t}$$

Implicit method: $$\sum_{\text{All sides}} \dot{Q}^{i+1} + \dot{G}^{\text{i+1}}_{\text{element}} = \rho V_{\text{element}} C \frac{T_m^{i+1} - T_m^i}{\Delta t}$$

5-65C The explicit finite difference formulation of a general interior node for transient heat conduction in a plane wall is given by $T_{m-1}^i - 2T_m^i + T_{m+1}^i + \frac{\dot{g}_m^i \Delta x^2}{k} = \frac{T_m^{i+1} - T_m^i}{\tau}$. The finite difference formulation for the steady case is obtained by simply setting $T_m^{i+1} = T_m^i$ and disregarding the time index *i*. It yields

$$T_{m-1} - 2T_m + T_{m+1} + \frac{\dot{g}_m \Delta x^2}{k} = 0$$

5-66C The explicit finite difference formulation of a general interior node for transient two-dimensional heat conduction is given by $T_{\text{node}}^{i+1} = \tau(T_{\text{left}}^{\text{i}} + T_{\text{top}}^{\text{i}} + T_{\text{right}}^{\text{i}} + T_{\text{bottom}}^{\text{i}}) + (1-4\tau)T_{\text{node}}^{\text{i}} + \tau \frac{\dot{g}_{\text{node}}^{\text{i}} l^2}{k}$. The finite difference formulation for the steady case is obtained by simply setting $T_m^{i+1} = T_m^i$ and disregarding the time index *i*. It yields

$$T_{\text{left}} + T_{\text{top}} + T_{\text{right}} + T_{\text{bottom}} - 4T_{\text{node}} + \frac{\dot{g}_{\text{node}} l^2}{k} = 0$$

5-67C There is a limitation on the size of the time step Δt in the solution of transient heat conduction problems using the explicit method, but there is no such limitation in the implicit method.

5-68C The general stability criteria for the explicit method of solution of transient heat conduction problems is expressed as follows: *The coefficients of all* T_m^i *in the* T_m^{i+1} *expressions* (*called the primary coefficient*) *in the simplified expressions must be greater than or equal to zero for all nodes m.*

5-69C For transient one-dimensional heat conduction in a plane wall with both sides of the wall at specified temperatures, the stability criteria for the explicit method can be expressed in its simplest form as

$$\tau = \frac{\alpha \Delta t}{(\Delta x)^2} \le \frac{1}{2}$$

5-70C For transient one-dimensional heat conduction in a plane wall with specified heat flux on both sides, the stability criteria for the explicit method can be expressed in its simplest form as

$$\tau = \frac{\alpha \Delta t}{(\Delta x)^2} \le \frac{1}{2}$$

which is identical to the one for the interior nodes. This is because the heat flux boundary conditions have no effect on the stability criteria.

5-71C For transient two-dimensional heat conduction in a rectangular region with insulation or specified temperature boundary conditions, the stability criteria for the explicit method can be expressed in its simplest form as

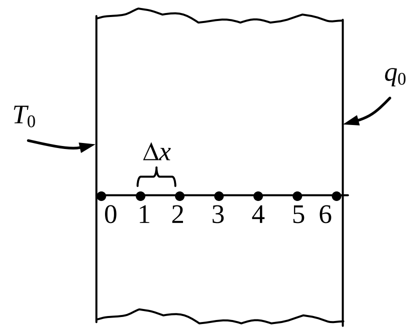
$$\tau = \frac{\alpha \Delta t}{(\Delta x)^2} \le \frac{1}{4}$$

which is identical to the one for the interior nodes. This is because the insulation or specified temperature boundary conditions have no effect on the stability criteria.

5-72C The implicit method is unconditionally stable and thus any value of time step Δt can be used in the solution of transient heat conduction problems since there is no danger of unstability. However, using a very large value of Δt is equivalent to replacing the time derivative by a very large difference, and thus the solution will not be accurate. Therefore, we should still use the smallest time step practical to minimize the numerical error.

5-73 A plane wall with no heat generation is subjected to specified temperature at the left (node 0) and heat flux at the right boundary (node 6). The explicit transient finite difference formulation of the boundary nodes and the finite difference formulation for the total amount of heat transfer at the left boundary during the first 3 time steps are to be determined.

Assumptions **1** Heat transfer through the wall is given to be transient, and the thermal conductivity to be constant. **2** Heat transfer is one-dimensional since the plate is large relative to its thickness. **3** There is no heat generation in the medium.



Analysis Using the energy balance approach and taking the direction of all heat transfers to be towards the node under consideration, the *explicit* finite difference formulations become

Left boundary node: $T_0^i = T_0 = 50°\text{C}$

Right boundary node: $$k\frac{T_5^i - T_6^i}{\Delta x} + \dot{q}_0 = \rho\frac{\Delta x}{2}C\frac{T_6^{i+1} - T_6^i}{\Delta t}$$

Heat transfer at left surface: $$\dot{Q}^i_{\text{left surface}} + kA\frac{T_1^i - T_0}{\Delta x} = \rho A\frac{\Delta x}{2}C\frac{T_6^{i+1} - T_6^i}{\Delta t}$$

Noting that $Q = \dot{Q}\Delta t = \sum_i \dot{Q}^i\Delta t$, the total amount of heat transfer becomes

$$Q_{\text{left surface}} = \sum_{i=1}^{3}\dot{Q}^i_{\text{left surface}}\Delta t = \sum_{i=1}^{3}\left(kA\frac{T_0 - T_1^i}{\Delta x} + A\frac{\Delta x}{2}C\frac{T_6^{i+1} - T_6^i}{\Delta t}\right)\Delta t$$

5-74 A plane wall with variable heat generation and constant thermal conductivity is subjected to uniform heat flux $\dot{q}_0$ at the left (node 0) and convection at the right boundary (node 4). The explicit transient finite difference formulation of the boundary nodes is to be determined.

Assumptions **1** Heat transfer through the wall is given to be transient, and the thermal conductivity to be constant. **2** Heat transfer is one-dimensional since the plate is large relative to its thickness. **3** Radiation heat transfer is negligible.

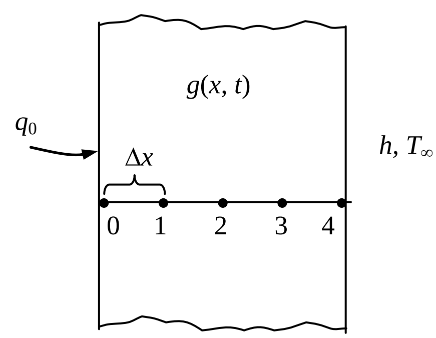
Analysis Using the energy balance approach and taking the direction of all heat transfers to be towards the node under consideration, the *explicit* finite difference formulations become

Left boundary node:

$$kA\frac{T_1^i - T_0^i}{\Delta x} + \dot{q}_0 A + \dot{g}_0^i(A\Delta x/2) = \rho A\frac{\Delta x}{2}C\frac{T_0^{i+1} - T_0^i}{\Delta t}$$

Right boundary node:

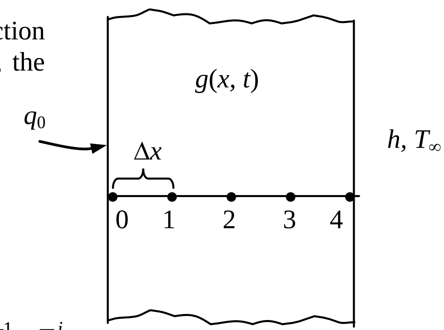
$$kA\frac{T_3^i - T_4^i}{\Delta x} + hA(T_\infty^i - T_4^i) + \dot{g}_4^i(A\Delta x/2) = \rho A\frac{\Delta x}{2}C\frac{T_4^{i+1} - T_4^i}{\Delta t}$$



5-75 A plane wall with variable heat generation and constant thermal conductivity is subjected to uniform heat flux $\dot{q}_0$ at the left (node 0) and convection at the right boundary (node 4). The explicit transient finite difference formulation of the boundary nodes is to be determined.

Assumptions **1** Heat transfer through the wall is given to be transient, and the thermal conductivity to be constant. **2** Heat transfer is one-dimensional since the plate is large relative to its thickness. **3** Radiation heat transfer is negligible.

Analysis Using the energy balance approach and taking the direction of all heat transfers to be towards the node under consideration, the *implicit* finite difference formulations become



Left boundary node:

$$kA\frac{T_1^{i+1}-T_0^{i+1}}{\Delta x}+\dot{q}_0A+\dot{g}_0^{i+1}(A\Delta x/2)=\rho A\frac{\Delta x}{2}C\frac{T_0^{i+1}-T_0^i}{\Delta t}$$

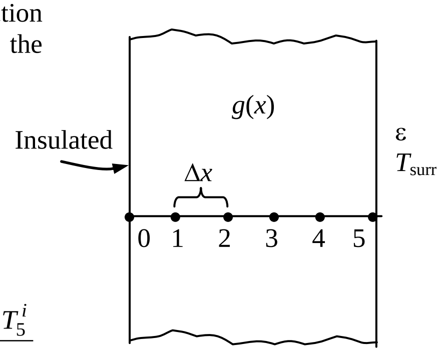
Right boundary node:

$$kA\frac{T_3^{i+1}-T_4^{i+1}}{\Delta x}+hA(T_\infty^{i+1}-T_4^{i+1})+\dot{g}_4^{i+1}(A\Delta x/2)=\rho A\frac{\Delta x}{2}C\frac{T_4^{i+1}-T_4^i}{\Delta t}$$

5-76 A plane wall with variable heat generation and constant thermal conductivity is subjected to insulation at the left (node 0) and radiation at the right boundary (node 5). The explicit transient finite difference formulation of the boundary nodes is to be determined.

Assumptions **1** Heat transfer through the wall is given to be transient and one-dimensional, and the thermal conductivity to be constant. **2** Convection heat transfer is negligible.

Analysis Using the energy balance approach and taking the direction of all heat transfers to be towards the node under consideration, the *explicit* transient finite difference formulations become



Left boundary node:

$$kA\frac{T_1^i-T_0^i}{\Delta x}+\dot{g}_0^iA\frac{\Delta x}{2}=\rho A\frac{\Delta x}{2}C\frac{T_0^{i+1}-T_0^i}{\Delta t}$$

Right boundary node:

$$\varepsilon\sigma A[(T_{\rm surr}^{\rm i})^4-(T_5^i)^4]+kA\frac{T_4^i-T_5^i}{\Delta x}+\dot{g}_5^iA\frac{\Delta x}{2}=\rho A\frac{\Delta x}{2}C\frac{T_5^{i+1}-T_5^i}{\Delta t}$$

5-77 A plane wall with variable heat generation and constant thermal conductivity is subjected to combined convection, radiation, and heat flux at the left (node 0) and specified temperature at the right boundary (node 4). The explicit finite difference formulation of the left boundary and the finite difference formulation for the total amount of heat transfer at the right boundary are to be determined.

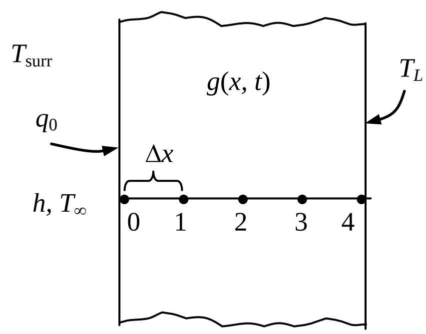
Assumptions **1** Heat transfer through the wall is given to be transient and one-dimensional, and the thermal conductivity to be constant. **2** Convection heat transfer is negligible.

Analysis Using the energy balance approach and taking the direction of all heat transfers to be towards the node under consideration, the *explicit* transient finite difference formulations become

Left boundary node: $\varepsilon\sigma A[T_{\text{surr}}^4 - (T_0^i)^4] + hA(T_\infty^i - T_0^i) + kA\dfrac{T_1^i - T_0^i}{\Delta x} + \dot{g}_0^i A\dfrac{\Delta x}{2} = \rho A\dfrac{\Delta x}{2}C\dfrac{T_0^{i+1} - T_0^i}{\Delta t}$

Heat transfer at right surface: $\dot{Q}_{\text{rightt surface}}^i + kA\dfrac{T_3^i - T_4^i}{\Delta x} + \dot{g}_4^i A\dfrac{\Delta x}{2} = \rho A\dfrac{\Delta x}{2}C\dfrac{T_4^{i+1} - T_4^i}{\Delta t}$

Noting that $Q = \dot{Q}\Delta t = \sum_i \dot{Q}^i \Delta t$, the total amount of heat transfer becomes



$$
\begin{aligned}
Q_{\text{right surface}} &= \sum_{i=1}^{20} \dot{Q}_{\text{right surface}}^i \Delta t \\
&= \sum_{i=1}^{20} \left(kA\frac{T_4^i - T_3^i}{\Delta x} - \dot{g}_4^i A\frac{\Delta x}{2} + \rho A\frac{\Delta x}{2}C\frac{T_4^{i+1} - T_4^i}{\Delta t} \right)\Delta t
\end{aligned}
$$

5-78 Starting with an energy balance on a volume element, the two-dimensional transient *explicit* finite difference equation for a general interior node in rectangular coordinates for $T(x, y, t)$ for the case of constant thermal conductivity and no heat generation is to be obtained.

Analysis (See Figure 5-49 in the text). We consider a rectangular region in which heat conduction is significant in the x and y directions, and consider a unit depth of $\Delta z = 1$ in the z direction. There is no heat generation in the medium, and the thermal conductivity k of the medium is constant. Now we divide the x-y plane of the region into a *rectangular mesh* of nodal points which are spaced Δx and Δy apart in the x and y directions, respectively, and consider a general interior node (m, n) whose coordinates are $x = m\Delta x$ and $y = n\Delta y$. Noting that the volume element centered about the general interior node (m, n) involves heat conduction from four sides (right, left, top, and bottom) and expressing them at previous time step i, the transient explicit finite difference formulation for a general interior node can be expressed as

$$k(\Delta y\times 1)\frac{T^i_{m-1,n}-T^i_{m,n}}{\Delta x}+k(\Delta x\times 1)\frac{T^i_{m,n+1}-T^i_{m,n}}{\Delta y}+k(\Delta y\times 1)\frac{T^i_{m+1,n}-T^i_{m,n}}{\Delta x}+k(\Delta x\times 1)\frac{T^i_{m,n-1}-T^i_{m,n}}{\Delta y}$$
$$=\rho(\Delta x\times\Delta y\times 1)C\,\frac{T^{i+1}_{m,n}-T^i_{m,n}}{\Delta t}$$

Taking a square mesh ($\Delta x = \Delta y = l$) and dividing each term by k gives, after simplifying,

$$T^i_{m-1,n}+T^i_{m+1,n}+T^i_{m,n+1}+T^i_{m,n-1}-4T^i_{m,n}=\frac{T^{i+1}_{m,n}-T^i_{m,n}}{\tau}$$

where $\alpha = k/(\rho C)$ is the thermal diffusivity of the material and $\tau = \alpha\Delta t/l^2$ is the dimensionless mesh Fourier number. It can also be expressed in terms of the temperatures at the neighboring nodes in the following easy-to-remember form:

$$T^i_{\text{left}}+T^i_{\text{top}}+T^i_{\text{right}}+T^i_{\text{bottom}}-4T^i_{\text{node}}=\frac{T^{i+1}_{\text{node}}-T^i_{\text{node}}}{\tau}$$

Discussion We note that setting $T^{i+1}_{\text{node}} = T^i_{\text{node}}$ gives the steady finite difference formulation.

5-79 Starting with an energy balance on a volume element, the two-dimensional transient *implicit* finite difference equation for a general interior node in rectangular coordinates for $T(x, y, t)$ for the case of constant thermal conductivity and no heat generation is to be obtained.

Analysis (See Figure 5-49 in the text). We consider a rectangular region in which heat conduction is significant in the x and y directions, and consider a unit depth of $\Delta z = 1$ in the z direction. There is no heat generation in the medium, and the thermal conductivity k of the medium is constant. Now we divide the x-y plane of the region into a *rectangular mesh* of nodal points which are spaced Δx and Δy apart in the x and y directions, respectively, and consider a general interior node (m, n) whose coordinates are $x = m\Delta x$ and $y = n\Delta y$. Noting that the volume element centered about the general interior node (m, n) involves heat conduction from four sides (right, left, top, and bottom) and expressing them at previous time step i, the transient *implicit* finite difference formulation for a general interior node can be expressed as

$$k(\Delta y \times 1)\frac{T_{m-1,n}^{i+1} - T_{m,n}^{i+1}}{\Delta x} + k(\Delta x \times 1)\frac{T_{m,n+1}^{i+1} - T_{m,n}^{i+1}}{\Delta y} + k(\Delta y \times 1)\frac{T_{m+1,n}^{i+1} - T_{m,n}^{i}}{\Delta x} + k(\Delta x \times 1)\frac{T_{m,n-1}^{i+1} - T_{m,n}^{i+1}}{\Delta y}$$
$$= \rho(\Delta x \times \Delta y \times 1)C\frac{T_{m,n}^{i+1} - T_{m,n}^{i}}{\Delta t}$$

Taking a square mesh ($\Delta x = \Delta y = l$) and dividing each term by k gives, after simplifying,

$$T_{m-1,n}^{i+1} + T_{m+1,n}^{i+1} + T_{m,n+1}^{i+1} + T_{m,n-1}^{i+1} - 4T_{m,n}^{i+1} = \frac{T_m^{i+1} - T_m^{i}}{\tau}$$

where $\alpha = k/(\rho C)$ is the thermal diffusivity of the material and $\tau = \alpha\Delta t / l^2$ is the dimensionless mesh Fourier number. It can also be expressed in terms of the temperatures at the neighboring nodes in the following easy-to-remember form:

$$T_{\text{left}}^{i+1} + T_{\text{top}}^{i+1} + T_{\text{right}}^{i+1} + T_{\text{bottom}}^{i+1} - 4T_{\text{node}}^{i+1} = \frac{T_{\text{node}}^{i+1} - T_{\text{node}}^{i}}{\tau}$$

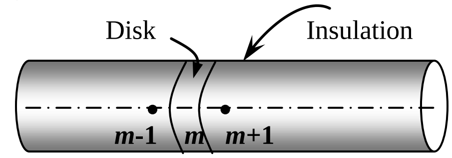
Discussion We note that setting $T_{\text{node}}^{i+1} = T_{\text{node}}^{i}$ gives the steady finite difference formulation.

5-80 Starting with an energy balance on a disk volume element, the one-dimensional transient explicit finite difference equation for a general interior node for $T(z,t)$ in a cylinder whose side surface is insulated for the case of constant thermal conductivity with uniform heat generation is to be obtained.

Analysis We consider transient one-dimensional heat conduction in the axial *z* direction in an insulated cylindrical rod of constant cross-sectional area *A* with constant heat generation $\dot{g}_0$ and constant conductivity *k* with a mesh size of Δz in the *z* direction. Noting that the volume element of a general interior node *m* involves heat conduction from two sides and the volume of the element is $V_{\text{element}} = A\Delta z$, the transient explicit finite difference formulation for an interior node can be expressed as

$$kA\frac{T_{m-1}^i - T_m^i}{\Delta x} + kA\frac{T_{m+1}^i - T_m^i}{\Delta x} + \dot{g}_0 A\Delta x = \rho A\Delta x C\frac{T_m^{i+1} - T_m^i}{\Delta t}$$

Canceling the surface area *A* and multiplying by *Δx*/k, it simplifies to



$$T_{m-1}^i - 2T_m^i + T_{m+1}^i + \frac{\dot{g}_0\Delta x^2}{k} = \frac{(\Delta x)^2}{\alpha\Delta t}(T_m^{i+1} - T_m^i)$$

where $\alpha = k/(\rho C)$ is the *thermal diffusivity* of the wall material.

Using the definition of the dimensionless *mesh Fourier number* $\tau = \dfrac{\alpha\Delta t}{(\Delta x)^2}$, the last equation reduces to

$$T_{m-1}^i - 2T_m^i + T_{m+1}^i + \frac{\dot{g}_0\Delta x^2}{k} = \frac{T_m^{i+1} - T_m^i}{\tau}$$

Discussion We note that setting $T_{\text{m}}^{i+1} = T_{\text{m}}^i$ gives the steady finite difference formulation.

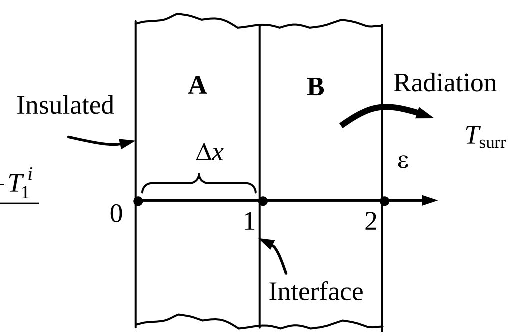
5-81 A composite plane wall consists of two layers A and B in perfect contact at the interface where node 1 is at the interface. The wall is insulated at the left (node 0) and subjected to radiation at the right boundary (node 2). The complete transient explicit finite difference formulation of this problem is to be obtained.

Assumptions **1** Heat transfer through the wall is given to be transient and one-dimensional, and the thermal conductivity to be constant. **2** Convection heat transfer is negligible. **3** There is no heat generation.

Analysis Using the energy balance approach with a unit area *A* = 1 and taking the direction of all heat transfers to be towards the node under consideration, the finite difference formulations become

Node 0 (at left boundary):

$$k_A\frac{T_1^i - T_0^i}{\Delta x} = \rho_A\frac{\Delta x}{2}C_A\frac{T_0^{i+1} - T_0^i}{\Delta t}$$



Node 1 (at interface):

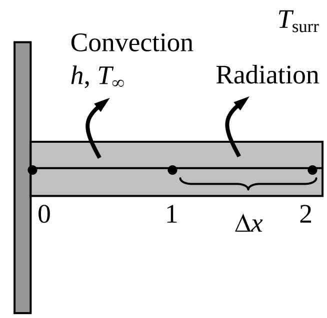
$$k_A\frac{T_0^i - T_1^i}{\Delta x} + k_B\frac{T_2^i - T_1^i}{\Delta x} = \left(\rho_A\frac{\Delta x}{2}C_A + \rho_B\frac{\Delta x}{2}C_B\right)\frac{T_1^{i+1} - T_1^i}{\Delta t}$$

Node 2 (at right boundary):

$$\varepsilon\sigma[T_{\text{surr}}^4 - (T_2^i)^4] + k_B\frac{T_1^i - T_2^i}{\Delta x} = \rho_B\frac{\Delta x}{2}C_B\frac{T_2^{i+1} - T_2^i}{\Delta t}$$

5-82 A pin fin with negligible heat transfer from its tip is considered. The complete finite difference formulation for the determination of nodal temperatures is to be obtained.

Assumptions **1** Heat transfer through the pin fin is given to be steady and one-dimensional, and the thermal conductivity to be constant. **2** Convection heat transfer coefficient is constant and uniform. **3** Heat loss from the fin tip is given to be negligible.



Analysis The nodal network consists of 3 nodes, and the base temperature T_0 at node 0 is specified. Therefore, there are two unknowns T_1 and T_2, and we need two equations to determine them. Using the energy balance approach and taking the direction of all heat transfers to be towards the node under consideration, the explicit transient finite difference formulations become

Node 1 (at midpoint):

$$\varepsilon\sigma p\Delta x[T_{surr}^4 - (T_1^i)^4] + hp\Delta x(T_\infty - T_1^i) + kA\frac{T_2^i - T_1^i}{\Delta x} + kA\frac{T_0^i - T_1^i}{\Delta x} = \rho A\Delta x C\frac{T_1^{i+1} - T_1^i}{\Delta t}$$

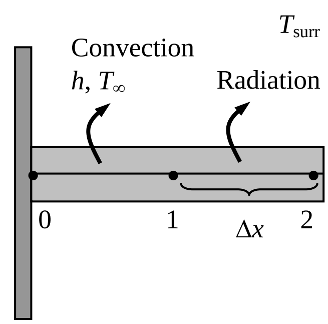
Node 2 (at fin tip):

$$\varepsilon\sigma\left(p\frac{\Delta x}{2}\right)[T_{surr}^4 - (T_2^i)^4] + h\left(p\frac{\Delta x}{2}\right)(T_\infty - T_2^i) + kA\frac{T_1^i - T_2^i}{\Delta x} = \rho A\frac{\Delta x}{2}C\frac{T_2^{i+1} - T_2^i}{\Delta t}$$

where $A = \pi D^2/4$ is the cross-sectional area and $p = \pi D$ is the perimeter of the fin.

5-83 A pin fin with negligible heat transfer from its tip is considered. The complete finite difference formulation for the determination of nodal temperatures is to be obtained.

Assumptions **1** Heat transfer through the pin fin is given to be steady and one-dimensional, and the thermal conductivity to be constant. **2** Convection heat transfer coefficient is constant and uniform. **3** Heat loss from the fin tip is given to be negligible.



Analysis The nodal network consists of 3 nodes, and the base temperature T_0 at node 0 is specified. Therefore, there are two unknowns T_1 and T_2, and we need two equations to determine them. Using the energy balance approach and taking the direction of all heat transfers to be towards the node under consideration, the implicit transient finite difference formulations become

Node 1: $$\varepsilon\sigma p\Delta x[T_{surr}^4 - (T_1^{i+1})^4] + hp\Delta x(T_\infty - T_1^{i+1}) + kA\frac{T_2^{i+1} - T_1^{i+1}}{\Delta x} + kA\frac{T_0^{i+1} - T_1^{i+1}}{\Delta x} = \rho A\Delta x C\frac{T_1^{i+1} - T_1^i}{\Delta t}$$

Node 2: $$\varepsilon\sigma\left(p\frac{\Delta x}{2}\right)[T_{surr}^4 - (T_2^{i+1})^4] + h\left(p\frac{\Delta x}{2}\right)(T_\infty - T_2^{i+1}) + kA\frac{T_1^{i+1} - T_2^{i+1}}{\Delta x} = \rho A\frac{\Delta x}{2}C\frac{T_2^{i+1} - T_2^i}{\Delta t}$$

where $A = \pi D^2/4$ is the cross-sectional area and $p = \pi D$ is the perimeter of the fin.