Special Topic: Controlling the Numerical Error

5-96C The results obtained using a numerical method differ from the exact results obtained analytically because the results obtained by a numerical method are approximate. The difference between a numerical solution and the exact solution (the error) is primarily due to two sources: The *discretization error* (also called the *truncation* or *formulation* error) which is caused by the approximations used in the formulation of the numerical method, and the *round-off error* which is caused by the computers' representing a number by using a limited number of significant digits and continuously rounding (or chopping) off the digits it cannot retain.

5-97C The *discretization error* (also called the *truncation* or *formulation* error) is due to replacing the derivatives by differences in each step, or replacing the actual temperature distribution between two adjacent nodes by a straight line segment. The difference between the two solutions at each time step is called the *local discretization error*. The total discretization error at any step is called the *global* or *accumulated discretization error*. The local and global discretization errors are identical for the first time step.

5-98C Yes, the global (accumulated) discretization error be less than the local error during a step. The global discretization error usually increases with increasing number of steps, but the opposite may occur when the solution function changes direction frequently, giving rise to local discretization errors of opposite signs which tend to cancel each other.

5-99C The Taylor series expansion of the temperature at a specified nodal point *m* about time t_i is

$$T(x_m, t_i + \Delta t) = T(x_m, t_i) + \Delta t \frac{\partial T(x_m, t_i)}{\partial t} + \frac{1}{2}\Delta t^2 \frac{\partial^2 T(x_m, t_i)}{\partial t^2} + \cdots$$

The finite difference formulation of the time derivative at the same nodal point is expressed as

$$\frac{\partial T(x_m, t_i)}{\partial t} \cong \frac{T(x_m, t_i + \Delta t) - T(x_m, t_i)}{\Delta t} = \frac{T_m^{i+1} - T_m^i}{\Delta t} \quad \text{or} \quad T(x_m, t_i + \Delta t) \cong T(x_m, t_i) + \Delta t \frac{\partial T(x_m, t_i)}{\partial t}$$

which resembles the Taylor series expansion terminated after the first two terms.

5-100C The Taylor series expansion of the temperature at a specified nodal point *m* about time t_i is

$$T(x_m, t_i + \Delta t) = T(x_m, t_i) + \Delta t \frac{\partial T(x_m, t_i)}{\partial t} + \frac{1}{2}\Delta t^2 \frac{\partial^2 T(x_m, t_i)}{\partial t^2} + \cdots$$

The finite difference formulation of the time derivative at the same nodal point is expressed as

$$\frac{\partial T(x_m, t_i)}{\partial t} \cong \frac{T(x_m, t_i + \Delta t) - T(x_m, t_i)}{\Delta t} = \frac{T_m^{i+1} - T_m^i}{\Delta t} \quad \text{or} \quad T(x_m, t_i + \Delta t) \cong T(x_m, t_i) + \Delta t \frac{\partial T(x_m, t_i)}{\partial t}$$

which resembles the Taylor series expansion terminated after the first two terms. Therefore, the 3rd and following terms in the Taylor series expansion represent the error involved in the finite difference approximation. For a sufficiently small time step, these terms decay rapidly as the order of derivative increases, and their contributions become smaller and smaller. The first term neglected in the Taylor series expansion is proportional to $(\Delta t)^2$, and thus the local discretization error is also proportional to $(\Delta t)^2$.

The global discretization error is proportional to the step size to Δt itself since, at the worst case, the accumulated discretization error after I time steps during a time period t_0 is $I\Delta t^2 = (t_0 / \Delta t)\Delta t^2 = t_0 \Delta t$ which is proportional to Δt.

5-101C The *round-off error* is caused by retaining a limited number of digits during calculations. It depends on the number of calculations, the method of rounding off, the type of the computer, and even the sequence of calculations. Calculations that involve the alternate addition of small and large numbers are most susceptible to round-off error.

5-102C As the step size is decreased, the discretization error decreases but the round-off error increases.

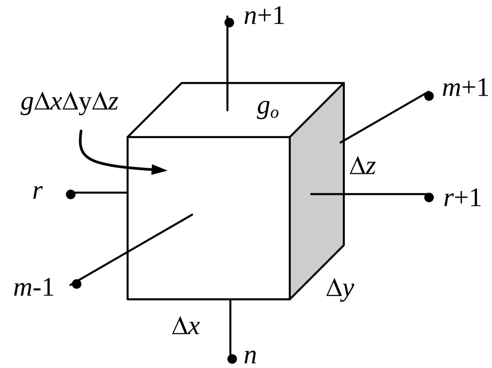
5-103C The round-off error can be reduced by avoiding extremely small mess sizes (smaller than necessary to keep the discretization error in check) and sequencing the terms in the program such that the addition of small and large numbers is avoided.

5-104C A practical way of checking if the round-off error has been significant in calculations is to repeat the calculations using double precision holding the mesh size and the size of the time step constant. If the changes are not significant, we conclude that the round-off error is not a problem.

5-105C A practical way of checking if the discretization error has been significant in calculations is to start the calculations with a reasonable mesh size Δx (and time step size Δt for transient problems), based on experience, and then to repeat the calculations using a mesh size of $\Delta x/2$. If the results obtained by halving the mesh size do not differ significantly from the results obtained with the full mesh size, we conclude that the discretization error is at an acceptable level.

Review Problems

5-106 Starting with an energy balance on a volume element, the steady three-dimensional finite difference equation for a general interior node in rectangular coordinates for $T(x, y, z)$ for the case of constant thermal conductivity and uniform heat generation is to be obtained.



Analysis We consider a *volume element* of size $\Delta x \times \Delta y \times \Delta z$ centered about a general interior node (m, n, r) in a region in which heat is generated at a constant rate of $\dot{g}_0$ and the thermal conductivity k is variable. Assuming the direction of heat conduction to be *towards* the node under consideration at all surfaces, the energy balance on the volume element can be expressed as

$$\dot{Q}_{\text{cond, left}} + \dot{Q}_{\text{cond, top}} + \dot{Q}_{\text{cond, right}} + \dot{Q}_{\text{cond, bottom}} + \dot{Q}_{\text{cond, front}} + \dot{Q}_{\text{cond, back}} + \dot{G}_{\text{element}} = \frac{\Delta E_{\text{element}}}{\Delta t} = 0$$

for the *steady* case. Again assuming the temperatures between the adjacent nodes to vary linearly, the energy balance relation above becomes

$$\begin{aligned} &k(\Delta y \times \Delta z)\frac{T_{m-1,n,r} - T_{m,n,r}}{\Delta x} + k(\Delta x \times \Delta z)\frac{T_{m,n+1,r} - T_{m,n,r}}{\Delta y} \\ &+ k(\Delta y \times \Delta z)\frac{T_{m+1,n,r} - T_{m,n,r}}{\Delta x} + k(\Delta x \times \Delta z)\frac{T_{m,n-1,r} - T_{m,n,r}}{\Delta y} \\ &+ k(\Delta x \times \Delta y)\frac{T_{m,n,r-1} - T_{m,n,r}}{\Delta z} + k(\Delta x \times \Delta y)\frac{T_{m,n,r+1} - T_{m,n,r}}{\Delta z} + \dot{g}_0(\Delta x \times \Delta y \times \Delta z) = 0 \end{aligned}$$

Dividing each term by $k\,\Delta x \times \Delta y \times \Delta z$ and simplifying gives

$$\frac{T_{m-1,n,r} - 2T_{m,n,r} + T_{m+1,n,r}}{\Delta x^2} + \frac{T_{m,n-1,r} - 2T_{m,n,r} + T_{m,n+1,r}}{\Delta y^2} + \frac{T_{m,n,r-1} - 2T_{m,n,r} + T_{m,n,r+1}}{\Delta z^2} + \frac{\dot{g}_0}{k} = 0$$

For a cubic mesh with $\Delta x = \Delta y = \Delta z = l$, and the relation above simplifies to

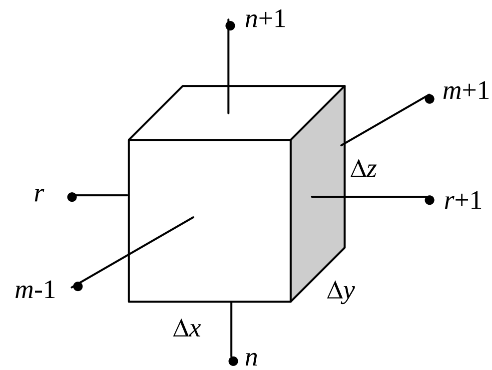
$$T_{m-1,n,r} + T_{m+1,n,r} + T_{m,n-1,r} + T_{m,n-1,r} + T_{m,n,r-1} + T_{m,n,r+1} - 6T_{m,n,r} + \frac{\dot{g}_0 l^2}{k} = 0$$

It can also be expressed in the following easy-to-remember form:

$$T_{\text{left}} + T_{\text{top}} + T_{\text{right}} + T_{\text{bottom}} + T_{\text{front}} + T_{\text{back}} - 6T_{\text{node}} + \frac{\dot{g}_0 l^2}{k} = 0$$

5-107 Starting with an energy balance on a volume element, the three-dimensional transient *explicit* finite difference equation for a general interior node in rectangular coordinates for $T(x, y, z, t)$ for the case of constant thermal conductivity k and no heat generation is to be obtained.

Analysis We consider a rectangular region in which heat conduction is significant in the x and y directions. There is no heat generation in the medium, and the thermal conductivity k of the medium is constant. Now we divide the x-y-z region into a *mesh* of nodal points which are spaced Δx, Δy, and Δz apart in the x, y, and z directions, respectively, and consider a general interior node (m, n, r) whose coordinates are $x = m\Delta x$, $y = n\Delta y$, are $z = r\Delta z$. Noting that the volume element centered about the general interior node (m, n, r) involves heat conduction from six sides (right, left, front, rear, top, and bottom) and expressing them at previous time step i, the transient explicit finite difference formulation for a general interior node can be expressed as



$$k(\Delta y\times\Delta z)\frac{T^i_{m-1,n,r}-T^i_{m,n,r}}{\Delta x}+k(\Delta x\times\Delta z)\frac{T^i_{m,n+1,r}-T^i_{m,n,r}}{\Delta y}+k(\Delta y\times\Delta z)\frac{T^i_{m+1,n,r}-T^i_{m,n,r}}{\Delta x}$$

$$+k(\Delta x\times\Delta z)\frac{T^i_{m,n-1,r}-T^i_{m,n,r}}{\Delta y}+k(\Delta x\times\Delta y)\frac{T^i_{m,n,r-1}-T^i_{m,n,r}}{\Delta z}+k(\Delta x\times\Delta y)\frac{T^i_{m,n,r+1}-T^i_{m,n,r}}{\Delta z}$$

$$=\rho(\Delta x\times\Delta y\times 1)C\,\frac{T^{i+1}_{m,n}-T^i_{m,n}}{\Delta t}$$

Taking a cubic mesh ($\Delta x = \Delta y = \Delta z = l$) and dividing each term by k gives, after simplifying,

$$T^i_{m-1,n,r}+T^i_{m+1,n,r}+T^i_{m,n+1,r}+T^i_{m,n-1,r}+T^i_{m,n,r-1}+T^i_{m,n,r+1}-6T^i_{m,n,r}=\frac{T^{i+1}_{m,n,r}-T^i_{m,n,r}}{\tau}$$

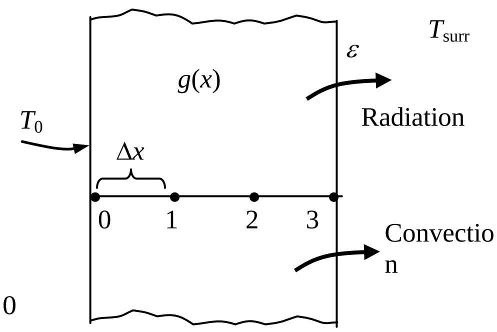
where $\alpha = k/(\rho C)$ is the thermal diffusivity of the material and $\tau = \alpha\Delta t/l^2$ is the dimensionless mesh Fourier number. It can also be expressed in terms of the temperatures at the neighboring nodes in the following easy-to-remember form:

$$T^i_{\text{left}}+T^i_{\text{top}}+T^i_{\text{right}}+T^i_{\text{bottom}}+T^i_{\text{front}}+T^i_{\text{back}}-6T^i_{\text{node}}=\frac{T^{i+1}_{\text{node}}-T^i_{\text{node}}}{\tau}$$

Discussion We note that setting $T^{i+1}_{\text{node}} = T^i_{\text{node}}$ gives the steady finite difference formulation.

5-108 A plane wall with variable heat generation and constant thermal conductivity is subjected to combined convection and radiation at the right (node 3) and specified temperature at the left boundary (node 0). The finite difference formulation of the right boundary node (node 3) and the finite difference formulation for the rate of heat transfer at the left boundary (node 0) are to be determined.

Assumptions **1** Heat transfer through the wall is given to be steady and one-dimensional. **2** The thermal conductivity is given to be constant.



Analysis Using the energy balance approach and taking the direction of all heat transfers to be towards the node under consideration, the finite difference formulations become

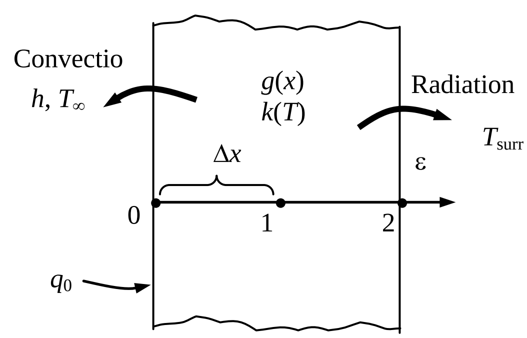
Right boundary node (all temperatures are in K):

$$\varepsilon\sigma A(T_{\rm surr}^4 - T_3^4) + hA(T_\infty - T_3) + kA\frac{T_2 - T_3}{\Delta x} + \dot{g}_3(A\Delta x/2) = 0$$

Heat transfer at left surface: $\dot{Q}_{\rm left\ surface} + kA\dfrac{T_1 - T_0}{\Delta x} + \dot{g}_0(A\Delta x/2) = 0$

5-109 A plane wall with variable heat generation and variable thermal conductivity is subjected to uniform heat flux $\dot{q}_0$ and convection at the left (node 0) and radiation at the right boundary (node 2). The explicit transient finite difference formulation of the problem using the energy balance approach method is to be determined.

Assumptions **1** Heat transfer through the wall is given to be transient, and the thermal conductivity and heat generation to be variables. **2** Heat transfer is one-dimensional since the plate is large relative to its thickness. **3** Radiation from the left surface, and convection from the right surface are negligible.



Analysis Using the energy balance approach and taking the direction of all heat transfers to be towards the node under consideration, the *explicit* finite difference formulations become

Left boundary node (node 0): $k_0^i A\dfrac{T_1^i - T_0^i}{\Delta x} + \dot{q}_0 A + hA(T_\infty - T_0^i) + \dot{g}_0^i(A\Delta x/2) = \rho A\dfrac{\Delta x}{2}C\dfrac{T_0^{i+1} - T_0^i}{\Delta t}$

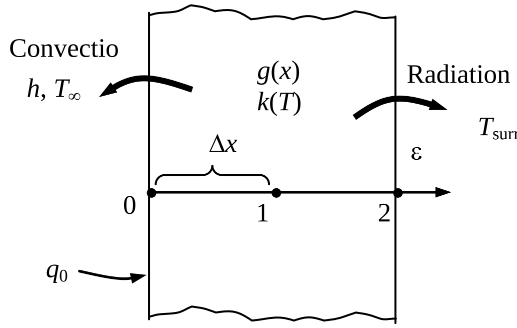
Interior node (node 1): $k_1^i A\dfrac{T_0^i - T_1^i}{\Delta x} + k_1^i A\dfrac{T_2^i - T_1^i}{\Delta x} + \dot{g}_0^i(A\Delta x) = \rho A\Delta x C\dfrac{T_1^{i+1} - T_1^i}{\Delta t}$

Right boundary node (node 2):

$$k_2^i A\frac{T_1^i - T_2^i}{\Delta x} + \varepsilon\sigma A[(T_{surr}^i + 273)^4 - (T_2^i + 273)^4] + \dot{g}_2^i(A\Delta x/2) = \rho A\frac{\Delta x}{2}C\frac{T_2^{i+1} - T_2^i}{\Delta t}$$

5-110 A plane wall with variable heat generation and variable thermal conductivity is subjected to uniform heat flux $\dot{q}_0$ and convection at the left (node 0) and radiation at the right boundary (node 2). The implicit transient finite difference formulation of the problem using the energy balance approach method is to be determined.

Assumptions **1** Heat transfer through the wall is given to be transient, and the thermal conductivity and heat generation to be variables. **2** Heat transfer is one-dimensional since the plate is large relative to its thickness. **3** Radiation from the left surface, and convection from the right surface are negligible.



Analysis Using the energy balance approach and taking the direction of all heat transfers to be towards the node under consideration, the *implicit* finite difference formulations become

Left boundary node (node 0):

$$k_0^{i+1}A\frac{T_1^{i+1}-T_0^{i+1}}{\Delta x}+\dot{q}_0A+hA(T_\infty-T_0^{i+1})+\dot{g}_0^{i+1}(A\Delta x/2)=\rho A\frac{\Delta x}{2}C\frac{T_0^{i+1}-T_0^i}{\Delta t}$$

Interior node (node 1): $$k_1^{i+1}A\frac{T_0^{i+1}-T_1^{i+1}}{\Delta x}+k_1^{i+1}A\frac{T_2^{i+1}-T_1^{i+1}}{\Delta x}+\dot{g}_0^{i+1}(A\Delta x)=\rho A\Delta xC\frac{T_1^{i+1}-T_1^i}{\Delta t}$$

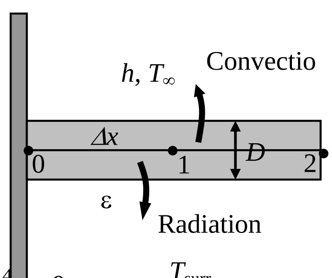
Right boundary node (node 2):

$$k_2^{i+1}A\frac{T_1^{i+1}-T_2^{i+1}}{\Delta x}+\varepsilon\sigma A[(T_{surr}^{i+1}+273)^4-(T_2^{i+1}+273)^4]+\dot{g}_2^{i+1}(A\Delta x/2)=\rho A\frac{\Delta x}{2}C\frac{T_2^{i+1}-T_2^i}{\Delta t}$$

5-111 A pin fin with convection and radiation heat transfer from its tip is considered. The complete finite difference formulation for the determination of nodal temperatures is to be obtained.

Assumptions **1** Heat transfer through the pin fin is given to be steady and one-dimensional, and the thermal conductivity to be constant. **2** Convection heat transfer coefficient and emissivity are constant and uniform.

Assumptions **1** Heat transfer through the wall is given to be steady and one-dimensional, and the thermal conductivity and heat generation to be variable. **2** Convection heat transfer at the right surface is negligible.



Analysis The nodal network consists of 3 nodes, and the base temperature T_0 at node 0 is specified. Therefore, there are two unknowns T_1 and T_2, and we need two equations to determine them. Using the energy balance approach and taking the direction of all heat transfers to be towards the node under consideration, the finite difference formulations become

Node 1 (at midpoint):

$$kA\frac{T_0-T_1}{\Delta x}+kA\frac{T_2-T_1}{\Delta x}+h(p\Delta x/2)(T_\infty-T_1)+\varepsilon\sigma A[T_{\text{surr}}^4-(T_1+273)^4]=0$$

Node 2 (at fin tip):

$$kA\frac{T_1-T_2}{\Delta x}+h(p\Delta x/2+A)(T_\infty-T_2)+\varepsilon\sigma(p\Delta x/2+A)[T_{\text{surr}}^4-(T_2+273)^4]=0$$

where $A=\pi D^2/4$ is the cross-sectional area and $p=\pi D$ is the perimeter of the fin.

5-112 Starting with an energy balance on a volume element, the two-dimensional transient *explicit* finite difference equation for a general interior node in rectangular coordinates for $T(x, y, t)$ for the case of constant thermal conductivity k and uniform heat generation $\dot{g}_0$ is to be obtained.

Analysis (See Figure 5-24 in the text)**.** We consider a rectangular region in which heat conduction is significant in the x and y directions, and consider a unit depth of $\Delta z = 1$ in the z direction. There is uniform heat generation in the medium, and the thermal conductivity k of the medium is constant. Now we divide the x-y plane of the region into a *rectangular mesh* of nodal points which are spaced Δx and Δy apart in the x and y directions, respectively, and consider a general interior node (m, n) whose coordinates are $x = m\Delta x$ and $y = n\Delta y$. Noting that the volume element centered about the general interior node (m, n) involves heat conduction from four sides (right, left, top, and bottom) and expressing them at previous time step i, the transient explicit finite difference formulation for a general interior node can be expressed as

$$k(\Delta y \times 1)\frac{T^i_{m-1,n} - T^i_{m,n}}{\Delta x} + k(\Delta x \times 1)\frac{T^i_{m,n+1} - T^i_{m,n}}{\Delta y} + k(\Delta y \times 1)\frac{T^i_{m+1,n} - T^i_{m,n}}{\Delta x} + k(\Delta x \times 1)\frac{T^i_{m,n-1} - T^i_{m,n}}{\Delta y}$$

$$+ \dot{g}_0(\Delta x \times \Delta y \times 1) = \rho(\Delta x \times \Delta y \times 1)C\frac{T^{i+1}_{m,n} - T^i_{m,n}}{\Delta t}$$

Taking a square mesh ($\Delta x = \Delta y = l$) and dividing each term by k gives, after simplifying,

$$T^i_{m-1,n} + T^i_{m+1,n} + T^i_{m,n+1} + T^i_{m,n-1} - 4T^i_{m,n} + \frac{\dot{g}_0 l^2}{k} = \frac{T^{i+1}_{m,n} - T^i_{m,n}}{\tau}$$

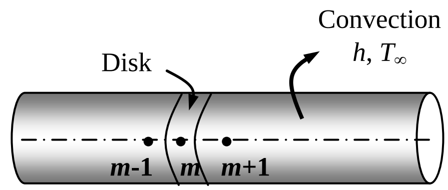
where $\alpha = k/(\rho C)$ is the thermal diffusivity of the material and $\tau = \alpha\Delta t / l^2$ is the dimensionless mesh Fourier number. It can also be expressed in terms of the temperatures at the neighboring nodes in the following easy-to-remember form:

$$T^i_{\text{left}} + T^i_{\text{top}} + T^i_{\text{right}} + T^i_{\text{bottom}} - 4T^i_{\text{node}} + \frac{\dot{g}_0 l^2}{k} = \frac{T^{i+1}_{\text{node}} - T^i_{\text{node}}}{\tau}$$

Discussion We note that setting $T^{i+1}_{\text{node}} = T^i_{\text{node}}$ gives the steady finite difference formulation.

5-113 Starting with an energy balance on a disk volume element, the one-dimensional transient explicit finite difference equation for a general interior node for $T(z,t)$ in a cylinder whose side surface is subjected to convection with a convection coefficient of h and an ambient temperature of T_∞ for the case of constant thermal conductivity with uniform heat generation is to be obtained.

Analysis We consider transient one-dimensional heat conduction in the axial z direction in a cylindrical rod of constant cross-sectional area A with constant heat generation $\dot{g}_0$ and constant conductivity k with a mesh size of Δz in the z direction. Noting that the volume element of a general interior node m involves heat conduction from two sides, convection from its lateral surface, and the volume of the element is $V_{\text{element}} = A\Delta z$, the transient explicit finite difference formulation for an interior node can be expressed as



$$hA(T_\infty - T_m^i) + kA\frac{T_{m-1}^i - T_m^i}{\Delta x} + kA\frac{T_{m+1}^i - T_m^i}{\Delta x} + \dot{g}_0 A\Delta x = \rho A\Delta x C\frac{T_m^{i+1} - T_m^i}{\Delta t}$$

Canceling the surface area A and multiplying by Δx/k, it simplifies to

$$T_{m-1}^i - (2 + h\Delta x/k)T_m^i + T_{m+1}^i + \frac{h\Delta x}{k}T_\infty + \frac{\dot{g}_0\Delta x^2}{k} = \frac{(\Delta x)^2}{\alpha\Delta t}(T_m^{i+1} - T_m^i)$$

where $\alpha = k/(\rho C)$ is the *thermal diffusivity* of the wall material. Using the definition of the dimensionless *mesh Fourier number* $\tau = \dfrac{\alpha\Delta t}{(\Delta x)^2}$ the last equation reduces to

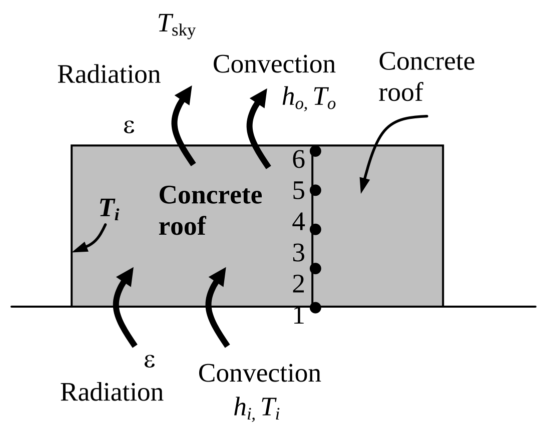
$$T_{m-1}^i - (2 + h\Delta x/k)T_m^i + T_{m+1}^i + \frac{h\Delta x}{k}T_\infty + \frac{\dot{g}_0\Delta x^2}{k} = \frac{T_m^{i+1} - T_m^i}{\tau}$$

Discussion We note that setting $T_{\text{m}}^{i+1} = T_{\text{m}}^i$ gives the steady finite difference formulation.

5-114E The roof of a house initially at a uniform temperature is subjected to convection and radiation on both sides. The temperatures of the inner and outer surfaces of the roof at 6 am in the morning as well as the average rate of heat transfer through the roof during that night are to be determined.

Assumptions **1** Heat transfer is one-dimensional since the roof is large relative to its thickness. **2** Thermal properties, heat transfer coefficients, and the indoor temperatures are constant. **3** Radiation heat transfer is significant. **4** The outdoor temperature remains constant in the 4-h blocks. **5** The given time step Δt = 5 min is less than the critical time step so that the stability criteria is satisfied.

Properties The conductivity and diffusivity are given to k = 0.81 Btu/h.ft.°F and $\alpha = 7.4\times10^{-6}\ \text{ft}^2/\text{s}$. The emissivity of both surfaces of the concrete roof is 0.9.



Analysis The nodal spacing is given to be Δx = 1 in. Then the number of nodes becomes $M = L/\Delta x + 1$ = 5/1+1 = 6. This problem involves 6 unknown nodal temperatures, and thus we need to have 6 equations. Nodes 2, 3, 4, and 5 are interior nodes, and thus for them we can use the general explicit finite difference relation expressed as

$$T_{m-1}^i - 2T_m^i + T_{m+1}^i + \frac{\dot{g}_m^i \Delta x^2}{k} = \frac{T_m^{i+1} - T_m^i}{\tau}$$

$$\rightarrow\ T_m^{i+1} = \tau(T_{m-1}^i + T_{m+1}^i) + (1-2\tau)T_m^i + \tau\frac{\dot{g}_m^i \Delta x^2}{k}$$

The finite difference equations for nodes 1 and 6 subjected to convection and radiation are obtained from an energy balance by taking the direction of all heat transfers to be towards the node under consideration:

$$\text{Node 1 (convection)}:\quad h_i(T_i - T_1^i) + k\frac{T_2^i - T_1^i}{\Delta x} + \varepsilon\sigma\left[T_{wall}^4 - (T_1^i + 273)^4\right] = \rho\frac{\Delta x}{2}C\frac{T_1^{i+1} - T_1^i}{\Delta t}$$

$$\text{Node 2 (interior)}:\quad T_2^{i+1} = \tau(T_1^i + T_3^i) + (1-2\tau)T_2^i$$

$$\text{Node 3 (interior)}:\quad T_3^{i+1} = \tau(T_2^i + T_4^i) + (1-2\tau)T_3^i$$

$$\text{Node 4 (interior)}:\quad T_4^{i+1} = \tau(T_3^i + T_5^i) + (1-2\tau)T_4^i$$

$$\text{Node 5 (interior)}:\quad T_5^{i+1} = \tau(T_4^i + T_6^i) + (1-2\tau)T_5^i$$

$$\text{Node 6 (convection)}:\quad h_o(T_0 - T_6^i) + k\frac{T_5^i - T_6^i}{\Delta x} + \varepsilon\sigma\left[T_{sky}^4 - (T_6^i + 273)^4\right] = \rho\frac{\Delta x}{2}C\frac{T_6^{i+1} - T_6^i}{\Delta t}$$

where k = 0.81 Btu/h.ft.°F, $\alpha = k/\rho C = 7.4\times10^{-6}\ \text{ft}^2/\text{s}$, T_i = 70°F, T_{wall} = 530 R, T_{sky} =445 R, h_i = 0.9 Btu/h.ft^2.°F, h_o = 2.1 Btu/h.ft^2.°F, Δx = 1/12 ft, and Δt = 5 min. Also, T_o = 50°F from 6 PM to 10 PM, 42°F from 10 PM to 2 AM, and 38°F from 2 AM to 6 AM. The mesh Fourier number is

$$\tau = \frac{\alpha\Delta t}{\Delta x^2} = \frac{(7.4\times10^{-6}\ \text{ft}^2/\text{s})(300\ \text{s})}{(1/12\ \text{ft})^2} = 0.320$$

Substituting this value of τ and other given quantities, the inner and outer surface temperatures of the roof after 12×(60/5) = 144 time steps (12 h) are determined to be **T_1 = 54.75°C and T_6 = 40.18°C**

(*b*) The average temperature of the inner surface of the roof can be taken to be

$$T_{1,ave} = \frac{T_{1\ @\ 6\ \text{PM}} + T_{1\ @\ 6\ \text{AM}}}{2} = \frac{70 + 54.75}{2} = 62.38°\text{F}$$

Then the average rate of heat loss through the roof that night is determined to be

$$\begin{aligned}\dot{Q}_{ave} &= h_i A(T_i - T_{1,ave}) + \varepsilon\sigma A\left[T_{wall}^4 - (T_1^i + 273)^4\right] \\ &= (0.9\ \text{Btu/h.ft}^2\cdot°\text{F})(45\times55\ \text{ft}^2)(70\text{-}62.38)°\text{F} \\ &\quad + 0.9(45\times55\ \text{ft}^2)(0.1714\times10^{-8}\ \text{Btu/h.ft}^2\cdot\text{R}^4)[(530\ \text{R})^4 - (62.38+460\ \text{R})^4] \\ &= \mathbf{33{,}950\ Btu/h}\end{aligned}$$

5-115 A large pond is initially at a uniform temperature. Solar energy is incident on the pond surface at for 4 h The temperature distribution in the pond under the most favorable conditions is to be determined.

Assumptions **1** Heat transfer is one-dimensional since the pond is large relative to its depth. **2** Thermal properties, heat transfer coefficients, and the indoor temperatures are constant. **3** Radiation heat transfer is significant. **4** There are no convection currents in the water. **5** The given time step Δt = 15 min is less than the critical time step so that the stability criteria is satisfied. **6** All heat losses from the pond are negligible. **7** Heat generation due to absorption of radiation is uniform in each layer.

Properties The conductivity and diffusivity are given to be k = 0.61 W/m.°C and $\alpha = 0.15\times 10^{-6}\ \text{m}^2/\text{s}$. The volumetric absorption coefficients of water are as given in the problem.

Analysis The nodal spacing is given to be Δx = 0.25 m. Then the number of nodes becomes $M = L/\Delta x + 1$ = 1/0.25+1 = 4. This problem involves 5 unknown nodal temperatures, and thus we need to have 5 equations. Nodes 2, 3, and 4 are interior nodes, and thus for them we can use the general explicit finite difference relation expressed as

$$T_{m-1}^i - 2T_m^i + T_{m+1}^i + \frac{\dot{g}_m^i \Delta x^2}{k} = \frac{T_m^{i+1} - T_m^i}{\tau} \;\rightarrow\; T_m^{i+1} = \tau(T_{m-1}^i + T_{m+1}^i) + (1-2\tau)T_m^i + \tau\frac{\dot{g}_m^i \Delta x^2}{k}$$

Node 0 can also be treated as an interior node by using the mirror image concept. The finite difference equation for node 4 subjected to heat flux is obtained from an energy balance by taking the direction of all heat transfers to be towards the node:

Node 0 (insulation): $T_0^{i+1} = \tau(T_1^i + T_1^i) + (1-2\tau)T_0^i + \tau\dot{g}_0(\Delta x)^2/k$

Node 0 (insulation): $T_1^{i+1} = \tau(T_0^i + T_2^i) + (1-2\tau)T_1^i + \tau\dot{g}_1(\Delta x)^2/k$

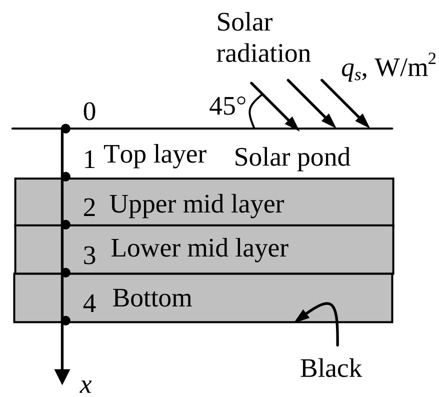
Node 2 (interior): $T_2^{i+1} = \tau(T_1^i + T_3^i) + (1-2\tau)T_2^i + \tau\dot{g}_2(\Delta x)^2/k$

Node 3 (interior): $T_3^{i+1} = \tau(T_2^i + T_4^i) + (1-2\tau)T_3^i + \tau\dot{g}_3(\Delta x)^2/k$

Node 6 (convection): $\dot{q}_b + k\dfrac{T_3^i - T_4^i}{\Delta x} + \tau\dot{g}_4(\Delta x)^2/k = \rho\dfrac{\Delta x}{2}C\dfrac{T_4^{i+1} - T_4^i}{\Delta t}$

where k = 0.61 W/m.°C, $\alpha = k/\rho C = 0.15\times 10^{-6}\ \text{m}^2/\text{s}$, Δx = 0.25 m, and Δt = 15 min = 900 s. Also, the mesh Fourier number is

$$\tau = \frac{\alpha\Delta t}{\Delta x^2} = \frac{(0.15\times 10^{-6}\ \text{m}^2/\text{s})(900\ \text{s})}{(0.25\ \text{m})^2} = 0.002160$$



The values of heat generation rates at the nodal points are determined as follows:

$$\dot{g}_0 = \frac{\dot{G}_0}{\text{Volume}} = \frac{0.473\times 500\ \text{W}}{(1\ \text{m}^2)(0.25\text{m})} = 946\ \text{W/m}^3$$

$$\dot{g}_1 = \frac{\dot{G}_1}{\text{Volume}} = \frac{[(0.473+0.061)/2]\times 500\ \text{W}}{(1\ \text{m}^2)(0.25\text{m})} = 534\ \text{W/m}^3$$

$$\dot{g}_4 = \frac{\dot{G}_4}{\text{Volume}} = \frac{0.024\times 500\ \text{W}}{(1\ \text{m}^2)(0.25\text{m})} = 48\ \text{W/m}^3$$

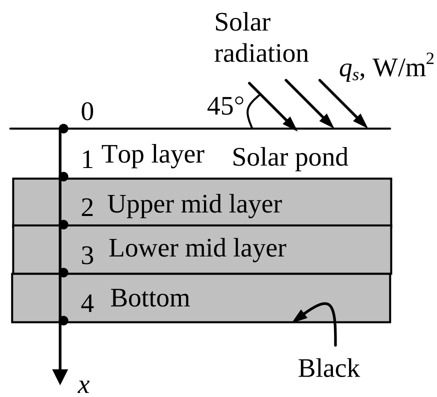
Also, the heat flux at the bottom surface is $\dot{q}_b = 0.379\times 500\ \text{W/m}^2 = 4189.5\ \text{W/m}^2$. Substituting these values, the nodal temperatures in the pond after 4×(60/15) = 16 time steps (4 h) are determined to be

T_0 = 18.3°C, T_1 = 16.9°C, T_2 = 15.4°C, T_3 = 15.3°C, and T_4 = 20.2°C.

5-116 A large 1-m deep pond is initially at a uniform temperature of 15°C throughout. Solar energy is incident on the pond surface at 45° at an average rate of 500 W/m^2 for a period of 4 h The temperature distribution in the pond under the most favorable conditions is to be determined.

Assumptions **1** Heat transfer is one-dimensional since the pond is large relative to its depth. **2** Thermal properties, heat transfer coefficients, and the indoor temperatures are constant. **3** Radiation heat transfer is significant. **4** There are no convection currents in the water. **5** The given time step Δt = 15 min is less than the critical time step so that the stability criteria is satisfied. **6** All heat losses from the pond are negligible. **7** Heat generation due to absorption of radiation is uniform in each layer.

Properties The conductivity and diffusivity are given to be k = 0.61 W/m.°C and $\alpha = 0.15\times10^{-6}\ \text{m}^2/\text{s}$. The volumetric absorption coefficients of water are as given in the problem.



Analysis The nodal spacing is given to be Δx = 0.25 m. Then the number of nodes becomes $M = L/\Delta x + 1$ = 1/0.25+1 = 4. This problem involves 5 unknown nodal temperatures, and thus we need to have 5 equations. Nodes 2, 3, and 4 are interior nodes, and thus for them we can use the general explicit finite difference relation expressed as

$$T_{m-1}^i - 2T_m^i + T_{m+1}^i + \frac{\dot{g}_m^i \Delta x^2}{k} = \frac{T_m^{i+1} - T_m^i}{\tau}$$

$$\rightarrow\ T_m^{i+1} = \tau(T_{m-1}^i + T_{m+1}^i) + (1-2\tau)T_m^i + \tau\frac{\dot{g}_m^i \Delta x^2}{k}$$

Node 0 can also be treated as an interior node by using the mirror image concept. The finite difference equation for node 4 subjected to heat flux is obtained from an energy balance by taking the direction of all heat transfers to be towards the node:

Node 0 (insulation): $\quad T_0^{i+1} = \tau(T_1^i + T_1^i) + (1-2\tau)T_0^i + \tau\dot{g}_0(\Delta x)^2/k$

Node 0 (insulation): $\quad T_1^{i+1} = \tau(T_0^i + T_2^i) + (1-2\tau)T_1^i + \tau\dot{g}_1(\Delta x)^2/k$

Node 2 (interior): $\quad T_2^{i+1} = \tau(T_1^i + T_3^i) + (1-2\tau)T_2^i + \tau\dot{g}_2(\Delta x)^2/k$

Node 3 (interior): $\quad T_3^{i+1} = \tau(T_2^i + T_4^i) + (1-2\tau)T_3^i + \tau\dot{g}_3(\Delta x)^2/k$

Node 6 (convection): $\quad \dot{q}_b + k\frac{T_3^i - T_4^i}{\Delta x} + \tau\dot{g}_4(\Delta x)^2/k = \rho\frac{\Delta x}{2}C\frac{T_4^{i+1} - T_4^i}{\Delta t}$

where k = 0.61 W/m.°C, $\alpha = k/\rho C = 0.15\times10^{-6}\ \text{m}^2/\text{s}$, Δx = 0.25 m, and Δt = 15 min = 900 s. Also, the mesh Fourier number is

$$\tau = \frac{\alpha\Delta t}{\Delta x^2} = \frac{(0.15\times10^{-6}\ \text{m}^2/\text{s})(900\ \text{s})}{(0.25\ \text{ft})^2} = 0.002160$$

The absorption of solar radiation is given to be $\dot{g}(x) = \dot{q}_s(0.859 - 3.415x + 6.704x^2 - 6.339x^3 + 2.278x^4)$ where $\dot{q}_s$ is the solar flux incident on the surface of the pond in W/m^2, and x is the distance form the free surface of the pond, in m. Then the values of heat generation rates at the nodal points are determined to be

Node 0 (x = 0): $\dot{g}_0 = 500(0.859 - 3.415\times0 + 6.704\times0^2 - 6.339\times0^3 + 2.278\times0^4) = 429.5\ \text{W/m}^3$

Node1(x=.25): $\dot{g}_1 = 500(0.859 - 3.415\times0.25 + 6.704\times0.25^2 - 6.339\times0.25^3 + 2.278\times0.25^4) = 167.1\ \text{W/m}^3$

Node 2 (x=0.50): $\dot{g}_2 = 500(0.859 - 3.415\times0.5 + 6.704\times0.5^2 - 6.339\times0.5^3 + 2.278\times0.5^4) = 88.8\ \text{W/m}^3$

Node3(x=.75): $\dot{g}_3 = 500(0.859 - 3.415\times0.75 + 6.704\times0.75^2 - 6.339\times0.75^3 + 2.278\times0.75^4) = 57.6\ \text{W/m}^3$

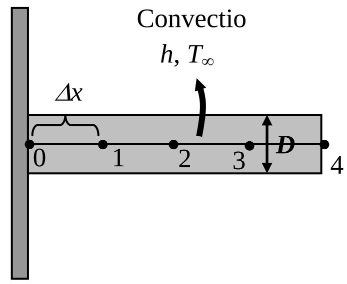
Node 4 (x = 1.00): $\dot{g}_4 = 500(0.859 - 3.415\times1 + 6.704\times1^2 - 6.339\times1^3 + 2.278\times1^4) = 43.5\ \text{W/m}^3$

Also, the heat flux at the bottom surface is $\dot{q}_b = 0.379\times500\ \text{W/m}^2 = 4189.5\ \text{W/m}^2$. Substituting these values, the nodal temperatures in the pond after 4×(60/15) = 16 time steps (4 h) are determined to be

T_0 = 16.5°C, T_1 = 15.6°C, T_2 = 15.3°C, T_3 = 15.3°C, and T_4 = 20.2°C.

5-117 A hot surface is to be cooled by aluminum pin fins. The nodal temperatures after 5 min are to be determined using the explicit finite difference method. Also to be determined is the time it takes for steady conditions to be reached.

Assumptions **1** Heat transfer through the pin fin is given to be one-dimensional. **2** The thermal properties of the fin are constant. **3** Convection heat transfer coefficient is constant and uniform. **4** Radiation heat transfer is negligible. **5** Heat loss from the fin tip is considered.



Analysis The nodal network of this problem consists of 5 nodes, and the base temperature T_0 at node 0 is specified. Therefore, there are 4 unknown nodal temperatures, and we need 4 equations to determine them. Using the energy balance approach and taking the direction of all heat transfers to be towards the node under consideration, the explicit transient finite difference formulations become

Node 1 (interior): $$hp\Delta x(T_\infty - T_1^i) + kA\frac{T_2^i - T_1^i}{\Delta x} + kA\frac{T_0 - T_1^i}{\Delta x} = \rho A\Delta x C\frac{T_1^{i+1} - T_1^i}{\Delta t}$$

Node 2 (interior): $$hp\Delta x(T_\infty - T_2^i) + kA\frac{T_3^i - T_2^i}{\Delta x} + kA\frac{T_1^i - T_2^i}{\Delta x} = \rho A\Delta x C\frac{T_2^{i+1} - T_2^i}{\Delta t}$$

Node 3 (interior): $$hp\Delta x(T_\infty - T_3^i) + kA\frac{T_4^i - T_3^i}{\Delta x} + kA\frac{T_2^i - T_3^i}{\Delta x} = \rho A\Delta x C\frac{T_3^{i+1} - T_3^i}{\Delta t}$$

Node 4 (fin tip): $$h(p\Delta x/2 + A)(T_\infty - T_4^i) + kA\frac{T_3^i - T_4^i}{\Delta x} = \rho A(\Delta x/2)C\frac{T_4^{i+1} - T_4^i}{\Delta t}$$

where $A = \pi D^2/4$ is the cross-sectional area and $p = \pi D$ is the perimeter of the fin. Also, D = 0.008 m, k = 237 W/m.°C, $\alpha = k/\rho C = 97.1\times10^{-6}\ \text{m}^2/\text{s}$, Δx = 0.02 m, T_∞ = 30°C, T_0 = 120°C h_o = 35 W/m^2.°C, and Δt = 1 s. Also, the mesh Fourier number is

$$\tau = \frac{\alpha\Delta t}{\Delta x^2} = \frac{(97.1\times10^{-6}\ \text{m}^2/\text{s})(1\ \text{s})}{(0.02\ \text{m})^2} = 0.24275$$

Substituting these values, the nodal temperatures along the fin after 5×60 = 300 time steps (4 h) are determined to be

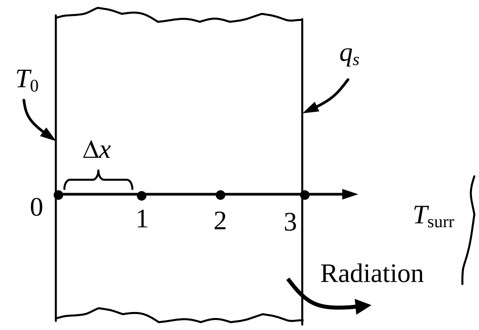
T_0 = 120°C, T_1 = 110.6°C, T_2 = 103.9°C, T_3 = 100.0°C, and T_4 = 98.5°C.

Printing the temperatures after each time step and examining them, we observe that the nodal temperatures stop changing after about 3.8 min. Thus we conclude that steady conditions are reached after **3.8 min.**

5-118E A plane wall in space is subjected to specified temperature on one side and radiation and heat flux on the other. The finite difference formulation of this problem is to be obtained, and the nodal temperatures under steady conditions are to be determined.

Assumptions **1** Heat transfer through the wall is given to be steady and one-dimensional. **2** Thermal conductivity is constant. **3** There is no heat generation. **4** There is no convection in space.

Properties The properties of the wall are given to be k=1.2 W/m·°C, ε = 0.80, and α_s = 0.45.



Analysis The nodal spacing is given to be Δx = 0.1 ft. Then the number of nodes becomes $M = L/\Delta x + 1$ = 0.3/0.1+1 = 4. The left surface temperature is given to be T_0 = 520 R = 60°F. This problem involves 3 unknown nodal temperatures, and thus we need to have 3 equations to determine them uniquely. Nodes 1 and 2 are interior nodes, and thus for them we can use the general finite difference relation expressed as

$$\frac{T_{m-1} - 2T_m + T_{m+1}}{\Delta x^2} + \frac{\dot{g}_m}{k} = 0 \rightarrow T_{m-1} - 2T_m + T_{m+1} = 0 \quad (\text{since } \dot{g} = 0), \quad \text{for } m = 1 \text{ and } 2$$

The finite difference equation for node 3 on the right surface subjected to convection and solar heat flux is obtained by applying an energy balance on the half volume element about node 3 and taking the direction of all heat transfers to be towards the node under consideration:

$$\text{Node 1 (interior):} \qquad T_0 - 2T_1 + T_2 = 0$$

$$\text{Node 2 (interior):} \qquad T_1 - 2T_2 + T_3 = 0$$

$$\text{Node 3 (right surface):} \qquad \alpha_s \dot{q}_s + \varepsilon\sigma[T_{\text{space}}^4 - (T_3 + 460)^4] + k\frac{T_2 - T_3}{\Delta x} = 0$$

where k = 1.2 Btu/h.ft.°F, ε = 0.80, α_s = 0.45, $\dot{q}_s = 300 \text{ Btu/h.ft}^2$, T_{space} = 0 R, and σ = 0.1714 Btu/h.ft^2.R^4 The system of 3 equations with 3 unknown temperatures constitute the finite difference formulation of the problem.

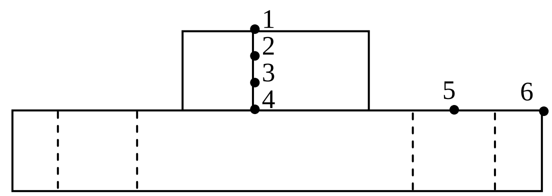
(*b*) The nodal temperatures under steady conditions are determined by solving the 3 equations above simultaneously with an equation solver to be

T_1 = 62.4°F = 522.4 R, T_2 = 64.8°F = 524.8 R, and T_3 = 67.3°F = 527.3 R

5-119 Frozen steaks are to be defrosted by placing them on a black-anodized circular aluminum plate. Using the explicit method, the time it takes to defrost the steaks is to be determined.

Assumptions **1** Heat transfer in both the steaks and the defrosting plate is one-dimensional since heat transfer from the lateral surfaces is negligible. **2** Thermal properties, heat transfer coefficients, and the surrounding air and surface temperatures remain constant during defrosting. **3** Heat transfer through the bottom surface of the plate is negligible. **4** The thermal contact resistance between the steaks and the plate is negligible. **5** Evaporation from the steaks and thus evaporative cooling is negligible. **6** The heat storage capacity of the plate is small relative to the amount of total heat transferred to the steak, and thus the heat transferred to the plate can be assumed to be transferred to the steak.

Properties The thermal properties of the steaks are ρ = 970 kg/m^3, C_p = 1.55 kJ/kg.°C, k = 1.40 W/m.°C, $\alpha = 0.93\times10^{-6}\ \text{m}^2/\text{s}$, ε = 0.95, and h_{if} = 187 kJ/kg. The thermal properties of the defrosting plate are k = 237 W/m.°C, $\alpha = 97.1\times10^{-6}\ \text{m}^2/\text{s}$, and ε = 0.90. The ρC_p (volumetric specific heat) values of the steaks and of the defrosting plate are



$$(\rho C_p)_{\text{plate}} = \frac{k}{\alpha} = \frac{237\ \text{W/m}\cdot°\text{C}}{97.1\times10^{-6}\ \text{m}^2/s} = 2441\text{kW/m}^3\cdot°\text{C}$$

$$(\rho C_p)_{\text{steak}} = (970\,\text{kg/m}^3)(1.55\text{kJ/kg}\cdot°\text{C}) = 1504\,\text{kW/m}^3\cdot°\text{C}$$

Analysis The nodal spacing is given to be Δx = 0.005 m in the steaks, and Δr = 0.0375 m in the plate. This problem involves 6 unknown nodal temperatures, and thus we need to have 6 equations. Nodes 2 and 3 are interior nodes in a plain wall, and thus for them we can use the general explicit finite difference relation expressed as

$$T_{m-1}^i - 2T_m^i + T_{m+1}^i + \frac{\dot{g}_m^i \Delta x^2}{k} = \frac{T_m^{i+1} - T_m^i}{\tau} \;\rightarrow\; T_m^{i+1} = \tau(T_{m-1}^i + T_{m+1}^i) + (1-2\tau)T_m^i$$

The finite difference equations for other nodes are obtained from an energy balance by taking the direction of all heat transfers to be towards the node under consideration:

Node 1: $$h(T_\infty - T_1^i) + \varepsilon_{\text{steak}}\sigma[(T_\infty + 273)^4 - (T_1^i + 273)^4] + k_{\text{steak}}\frac{T_2^i - T_1^i}{\Delta x} = (\rho C)_{\text{steak}}\frac{\Delta x}{2}\frac{T_1^{i+1} - T_1^i}{\Delta t}$$

Node 2 (interior): $$T_2^{i+1} = \tau_{\text{steak}}(T_1^i + T_3^i) + (1 - 2\tau_{\text{steak}})T_2^i$$

Node 3 (interior): $$T_3^{i+1} = \tau_{\text{steak}}(T_2^i + T_4^i) + (1 - 2\tau_{\text{steak}})T_3^i$$

Node 4:

$$\pi(r_{45}^2 - r_4^2)\{h(T_\infty - T_4^i) + \varepsilon_{\text{plate}}\sigma[(T_\infty + 273)^4 - (T_4^i + 273)^4]\} + k_{\text{steak}}(\pi r_4^2)\frac{T_3^i - T_4^i}{\Delta x}$$
$$+ k_{\text{plate}}(2\pi r_{45}\delta)\frac{T_5^i - T_4^i}{\Delta r} = [(\rho C)_{\text{steak}}(\pi r_4^2 \Delta x/2) + (\rho C)_{\text{plate}}(\pi r_{45}^2\delta)]\frac{T_4^{i+1} - T_4^i}{\Delta t}$$

Node 5:

$$2\pi r_5 \Delta r\{h(T_\infty - T_5^i) + \varepsilon_{\text{plate}}\sigma[(T_\infty + 273)^4 - (T_5^i + 273)^4]\}$$
$$+ k_{\text{plate}}(2\pi r_{56}\delta)\frac{T_6^i - T_5^i}{\Delta r} = (\rho C)_{\text{plate}}(\pi r_5^2\delta)]\frac{T_5^{i+1} - T_5^i}{\Delta t}$$

Node 6:

$$2\pi[(r_{56}+r_6)/2](\Delta r/2)\{h(T_\infty - T_6^i)+\varepsilon_{\text{plate}}\sigma[(T_\infty+273)^4-(T_6^i+273)^4]\}$$
$$+k_{\text{plate}}(2\pi r_{56}\delta)\frac{T_5^i-T_6^i}{\Delta r}=(\rho C)_{\text{plate}}[2\pi(r_{56}+r_6)/2](\Delta r/2)\delta\frac{T_6^{i+1}-T_6^i}{\Delta t}$$

where $(\rho C_p)_{\text{plate}} = 2441\,\text{kW/m}^3\cdot°\text{C}$, $(\rho C_p)_{\text{steak}} = 1504\,\text{kW/m}^3\cdot°\text{C}$, k_{steak} = 1.40 W/m.°C, , $\varepsilon_{\text{steak}}$ = 0.95, $\alpha_{\text{steak}} = 0.93\times10^{-6}\ \text{m}^2/\text{s}$, h_{if} = 187 kJ/kg, k_{plate} = 237 W/m.°C, $\alpha_{\text{plate}} = 97.1\times10^{-6}\ \text{m}^2/\text{s}$, and $\varepsilon_{\text{plate}}$ = 0.90, T_∞ = 20°C, h =12 W/m^2.°C, δ = 0.01 m, Δx = 0.005 m, Δr = 0.0375 m, and Δt = 5 s. Also, the mesh Fourier number for the steaks is

$$\tau_{\text{steak}}=\frac{\alpha\Delta t}{\Delta x^2}=\frac{(0.93\times10^{-6}\ \text{m}^2/\text{s})(5\ \text{s})}{(0.005\ \text{m})^2}=0.186$$

The various radii are r_4 =0.075 m, r_5 =0.1125 m, r_6 =0.15 m, r_{45} = (0.075+0.1125)/2 m, and r_{56} = (0.1125+0.15)/2 m.

The total amount of heat transfer needed to defrost the steaks is

$$m_{\text{steak}}=\rho V=(970\ \text{kg/m}^3)[\pi(0.075\ \text{m})^2(0.015\ \text{m})]=0.257\ \text{kg}$$

$$Q_{\text{total, steak}}=Q_{\text{sensible}}+Q_{\text{latent}}=(mC\Delta T)_{\text{steak}}+(mh_{\text{if}})_{\text{steak}}$$
$$=(0.257\ \text{kg})(1.55\ \text{kJ/kg.°C})[0-(-18°\text{C})]+(0.257\ \text{kg})(187\ \text{kJ/kg})=55.2\ \text{kJ}$$

The amount of heat transfer to the steak during a time step i is the sum of the heat transferred to the steak directly from its top surface, and indirectly through the plate, and is expressed as

$$Q_{\text{steak}}^i=2\pi r_5\Delta r\{h(T_\infty-T_5^i)+\varepsilon_{\text{plate}}\sigma[(T_\infty+273)^4-(T_5^i+273)^4]\}$$
$$+2\pi[(r_{56}+r_6)/2](\Delta r/2)\{h(T_\infty-T_6^i)+\varepsilon_{\text{plate}}\sigma[(T_\infty+273)^4-(T_6^i+273)^4]\}$$
$$+\pi(r_{45}^2-r_4^2)\{h(T_\infty-T_4^i)+\varepsilon_{\text{plate}}\sigma[(T_\infty+273)^4-(T_4^i+273)^4]\}$$
$$+\pi r_1^2\{h(T_\infty-T_1^i)+\varepsilon_{\text{steak}}\sigma[(T_\infty+273)^4-(T_1^i+273)^4]\}$$

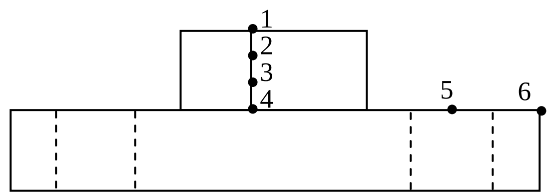
The defrosting time is determined by finding the amount of heat transfer during each time step, and adding them up until we obtain 55.2 kJ. Multiplying the number of time steps N by the time step Δt = 5 s will give the defrosting time. In this case it is determined to be

$\Delta t_{\text{defrost}}$ = NΔt = 44(5 s) = **220 s**

5-120 Frozen steaks at -18°C are to be defrosted by placing them on a 1-cm thick black-anodized circular copper defrosting plate. Using the explicit finite difference method, the time it takes to defrost the steaks is to be determined.

Assumptions **1** Heat transfer in both the steaks and the defrosting plate is one-dimensional since heat transfer from the lateral surfaces is negligible. **2** Thermal properties, heat transfer coefficients, and the surrounding air and surface temperatures remain constant during defrosting. **3** Heat transfer through the bottom surface of the plate is negligible. **4** The thermal contact resistance between the steaks and the plate is negligible. **5** Evaporation from the steaks and thus evaporative cooling is negligible. **6** The heat storage capacity of the plate is small relative to the amount of total heat transferred to the steak, and thus the heat transferred to the plate can be assumed to be transferred to the steak.

Properties The thermal properties of the steaks are ρ = 970 kg/m^3, C_p = 1.55 kJ/kg.°C, k = 1.40 W/m.°C, $\alpha = 0.93\times10^{-6}\ \text{m}^2/\text{s}$, ε = 0.95, and h_{if} = 187 kJ/kg. The thermal properties of the defrosting plate are k = 401 W/m.°C, $\alpha = 117\times10^{-6}\ \text{m}^2/\text{s}$, and ε = 0.90 (Table A-3). The ρC_p (volumetric specific heat) values of the steaks and of the defrosting plate are



$$(\rho C_p)_{\text{plate}} = (8933\,\text{kg/m}^3)(0.385\,\text{kJ/kg}\cdot°\text{C}) = 3439\,\text{kW/m}^3\cdot°\text{C}$$
$$(\rho C_p)_{\text{steak}} = (970\,\text{kg/m}^3)(1.55\,\text{kJ/kg}\cdot°\text{C}) = 1504\,\text{kW/m}^3\cdot°\text{C}$$

Analysis The nodal spacing is given to be Δx = 0.005 m in the steaks, and Δr = 0.0375 m in the plate. This problem involves 6 unknown nodal temperatures, and thus we need to have 6 equations. Nodes 2 and 3 are interior nodes in a plain wall, and thus for them we can use the general explicit finite difference relation expressed as

$$T_{m-1}^i - 2T_m^i + T_{m+1}^i + \frac{\dot{g}_m^i \Delta x^2}{k} = \frac{T_m^{i+1} - T_m^i}{\tau} \;\rightarrow\; T_m^{i+1} = \tau(T_{m-1}^i + T_{m+1}^i) + (1-2\tau)T_m^i$$

The finite difference equations for other nodes are obtained from an energy balance by taking the direction of all heat transfers to be towards the node under consideration:

Node 1: $$h(T_\infty - T_1^i) + \varepsilon_{\text{steak}}\sigma[(T_\infty + 273)^4 - (T_1^i + 273)^4] + k_{\text{steak}}\frac{T_2^i - T_1^i}{\Delta x} = (\rho C)_{\text{steak}}\frac{\Delta x}{2}\frac{T_1^{i+1} - T_1^i}{\Delta t}$$

Node 2 (interior): $$T_2^{i+1} = \tau_{\text{steak}}(T_1^i + T_3^i) + (1 - 2\tau_{\text{steak}})T_2^i$$
Node 3 (interior): $$T_3^{i+1} = \tau_{\text{steak}}(T_2^i + T_4^i) + (1 - 2\tau_{\text{steak}})T_3^i$$

Node 4:

$$\pi(r_{45}^2 - r_4^2)\{h(T_\infty - T_4^i) + \varepsilon_{\text{plate}}\sigma[(T_\infty + 273)^4 - (T_4^i + 273)^4]\} + k_{\text{steak}}(\pi r_4^2)\frac{T_3^i - T_4^i}{\Delta x}$$
$$+ k_{\text{plate}}(2\pi r_{45}\delta)\frac{T_5^i - T_4^i}{\Delta r} = [(\rho C)_{\text{steak}}(\pi r_4^2 \Delta x/2) + (\rho C)_{\text{plate}}(\pi r_{45}^2\delta)]\frac{T_4^{i+1} - T_4^i}{\Delta t}$$

Node 5:

$$2\pi r_5 \Delta r\{h(T_\infty - T_5^i) + \varepsilon_{\text{plate}}\sigma[(T_\infty + 273)^4 - (T_5^i + 273)^4]\}$$
$$+ k_{\text{plate}}(2\pi r_{56}\delta)\frac{T_6^i - T_5^i}{\Delta r} = (\rho C)_{\text{plate}}(\pi r_5^2\delta)]\frac{T_5^{i+1} - T_5^i}{\Delta t}$$

Node 6:

$$2\pi[(r_{56}+r_6)/2](\Delta r/2)\{h(T_\infty - T_6^i)+\varepsilon_{\text{plate}}\sigma[(T_\infty+273)^4-(T_6^i+273)^4]\}$$
$$+k_{\text{plate}}(2\pi r_{56}\delta)\frac{T_5^i-T_6^i}{\Delta r}=(\rho C)_{\text{plate}}[2\pi(r_{56}+r_6)/2](\Delta r/2)\delta\frac{T_6^{i+1}-T_6^i}{\Delta t}$$

where $(\rho C_p)_{\text{plate}}=3439\,\text{kW/m}^3\cdot{}^\circ\text{C}$, $(\rho C_p)_{\text{steak}}=1504\,\text{kW/m}^3\cdot{}^\circ\text{C}$, k_{steak} = 1.40 W/m.°C, $\varepsilon_{\text{steak}}$ = 0.95, $\alpha_{\text{steak}}=0.93\times10^{-6}\ \text{m}^2/\text{s}$, h_{if} = 187 kJ/kg, k_{plate} = 401 W/m.°C, $\alpha_{\text{plate}}=117\times10^{-6}\ \text{m}^2/\text{s}$, and $\varepsilon_{\text{plate}}$ = 0.90, T_∞ = 20°C, h =12 W/m^2.°C, δ = 0.01 m, Δx = 0.005 m, Δr = 0.0375 m, and Δt = 5 s. Also, the mesh Fourier number for the steaks is

$$\tau_{\text{steak}}=\frac{\alpha\Delta t}{\Delta x^2}=\frac{(0.93\times10^{-6}\ \text{m}^2/\text{s})(5\ \text{s})}{(0.005\ \text{m})^2}=0.186$$

The various radii are r_4 =0.075 m, r_5 =0.1125 m, r_6 =0.15 m, r_{45} = (0.075+0.1125)/2 m, and r_{56} = (0.1125+0.15)/2 m.

The total amount of heat transfer needed to defrost the steaks is

$$m_{\text{steak}}=\rho V=(970\ \text{kg/m}^3)[\pi(0.075\ \text{m})^2(0.015\ \text{m})]=0.257\ \text{kg}$$

$$\begin{aligned}Q_{\text{total, steak}}&=Q_{\text{sensible}}+Q_{\text{latent}}=(mC\Delta T)_{\text{steak}}+(mh_{\text{if}})_{\text{steak}}\\&=(0.257\ \text{kg})(1.55\ \text{kJ/kg.}^\circ\text{C})[0-(-18^\circ\text{C})]+(0.257\ \text{kg})(187\ \text{kJ/kg})=55.2\ \text{kJ}\end{aligned}$$

The amount of heat transfer to the steak during a time step i is the sum of the heat transferred to the steak directly from its top surface, and indirectly through the plate, and is expressed as

$$\begin{aligned}Q_{\text{steak}}^i&=2\pi r_5\Delta r\{h(T_\infty-T_5^i)+\varepsilon_{\text{plate}}\sigma[(T_\infty+273)^4-(T_5^i+273)^4]\}\\&+2\pi[(r_{56}+r_6)/2](\Delta r/2)\{h(T_\infty-T_6^i)+\varepsilon_{\text{plate}}\sigma[(T_\infty+273)^4-(T_6^i+273)^4]\}\\&+\pi(r_{45}^2-r_4^2)\{h(T_\infty-T_4^i)+\varepsilon_{\text{plate}}\sigma[(T_\infty+273)^4-(T_4^i+273)^4]\}\\&+\pi r_1^2\{h(T_\infty-T_1^i)+\varepsilon_{\text{steak}}\sigma[(T_\infty+273)^4-(T_1^i+273)^4]\}\end{aligned}$$

The defrosting time is determined by finding the amount of heat transfer during each time step, and adding them up until we obtain 55.2 kJ. Multiplying the number of time steps N by the time step Δt = 5 s will give the defrosting time. In this case it is determined to be

$\Delta t_{\text{defrost}}$ = NΔt = 47(5 s) = **235 s**

5-121 ….. 5-124 Design and Essay Problems
