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CHAPTER 7
PROJECTILES

7.1 *No Air Resistance*

We suppose that a particle is projected from a point O at the origin of a coordinate system, the y-axis being vertical and the x-axis directed along the ground. The particle is projected in the xy-plane, with initial speed V_0 at an angle α to the horizon. At any subsequent time in its motion its speed is V and the angle that its motion makes with the horizontal is ψ.

The initial horizontal component if the velocity is $V_0 \cos \alpha$, and, in the absence of air resistance, this horizontal component remains constant throughout the motion. I shall also refer to this constant horizontal component of the velocity as u. I.e. $u = V_0 \cos \alpha$ = constant throughout the motion.

The initial vertical component of the velocity is $V_0 \sin \alpha$, but the vertical component of the motion is decelerated at a constant rate g. At a later time during the motion, the vertical component of the velocity is $V \sin \psi$, which I shall also refer to as v.

In the following, I write in the left hand column the horizontal component of the equation of motion and the first and second time integrals; in the right hand column I do the same for the vertical component.

Horizontal.	Vertical	
$\ddot{x} = 0$	$\ddot{y} = -g$	7.1.1*a,b*
$\dot{x} = u = V_0 \cos \alpha$	$\dot{y} = v = V_0 \sin \alpha - gt$	7.1.2*a,b*
$x = V_0 t \cos \alpha$	$y = V_0 t \sin \alpha - \frac{1}{2} g t^2$	7.1.3*a,b*

The two equations 7.1.3*a,b* are the parametric equations to the trajectory. In vector form, these two equations could be written as a single vector equation:

$$\mathbf{r} = \mathbf{V_0} t + \tfrac{1}{2} \mathbf{g} t^2 . \qquad 7.1.4$$

Note the + sign on the right hand side of equation 7.1.4. The vector **g** is directed downwards.

The xy-equation to the trajectory is found by eliminating t between equations 7.1.3*a* and 7.1.3*b* to yield:

$$y = x \tan \alpha - \frac{g x^2}{2 V_0^2 \cos^2 \alpha} . \qquad 7.1.5$$

Now, re-write this in the form

$$x^2 - (\ \ )x = -(\ \ )y.$$

Add to each side (half the coefficient of x)^2 in order to "complete the square" on the left hand side, and, after some algebra, it will be found that the equation to the trajectory can be written as:

$$(x-A)^2 = -4a(y-B), \qquad 7.1.6$$

where
$$A = \frac{V_0^2 \sin\alpha\cos\alpha}{g} = \frac{V_0^2 \sin 2\alpha}{2g}, \qquad 7.1.7$$

$$B = \frac{V_0^2 \sin^2\alpha}{2g}, \qquad 7.1.8$$

and
$$a = \frac{V_0^2 \cos^2\alpha}{2g}. \qquad 7.1.9$$

Having re-arranged equation 7.1.5 in the form 7.1.6, we see that the trajectory is a *parabola* whose vertex is at (A , B). The *range on the horizontal plane* is 2A, or $\frac{V_0^2 \sin 2\alpha}{g}$. The greatest range on the horizontal plane is obtained when $\sin 2\alpha = 1$, or $\alpha = 45^\circ$. The greatest range on the horizontal plane is therefore V_0^2/g. The *maximum height* reached is B, or $\frac{V_0^2 \sin^2\alpha}{2g}$. The *distance between vertex and focus* is a, or $\frac{V_0^2 \cos^2\alpha}{2g}$. The focus is above ground if this is less than the maximum height, and below ground if it is greater than the maximum height. That is, the focus is above ground if $\cos^2\alpha < \sin^2\alpha$. That is to say, the focus is above ground if $\alpha > 45^\circ$ and below ground if $\alpha < 45^\circ$.

The radius of curvature ρ anywhere along the trajectory can be found using the usual formula $\rho = \frac{(1+y'^2)^{\frac{3}{2}}}{y''}$. At the top of the trajectory, $y' = 0$, so that $\rho = 1/y'$. Alternatively (in case one has forgotten or is unfamiliar with the "usual formula"), we note that the speed at the top of the path is just equal to the (constant) horizontal component of the velocity $V_0 \cos\alpha$. We can then equate the centripetal acceleration $(V_0^2\cos^2\alpha/\rho)$ to g and hence obtain:

$$\rho = \frac{V_0^2\cos^2\alpha}{g}. \qquad 7.1.10$$

By subtracting this from our expression for the maximum height of the projectile, we find that the height of the centre of curvature above the ground is $\frac{V_0^2(1-3\cos^2\alpha)}{2g}$. The centre of curvature is above ground if $\alpha > 54^\circ\ 44'$.

The *range r on a plane inclined at an angle q to the horizontal* can be found by substituting $x = r\cos\theta$ and $y = r\sin\theta$ in the equation 7.1.5 to the trajectory. This results, after some algebra, in

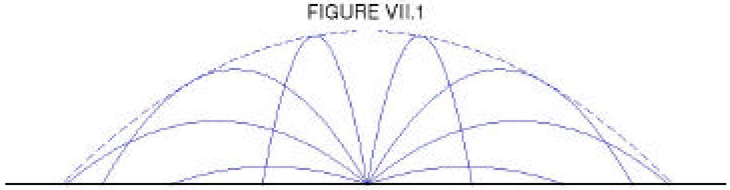
$$r = \frac{V_0^2}{g\cos^2\theta}\left[\sin(2\alpha-\theta)-\sin\theta\right]. \qquad 7.1.11$$

This is greatest when $2\alpha - \theta = 90^\circ$; *i.e.* when the angle of projection bisects the angle between the inclined plane and the vertical. The maximum range is

$$r = \frac{V_0^2}{g(1+\sin\theta)}. \qquad 7.1.12$$

This is the equation, in polar coordinates, of a *parabola*, and this parabola, when rotated about its vertical axis, describes a *paraboloid*, known as the *paraboloid of safety*. It is the envelope of all possible trajectories with an initial speed V_0. If a gun is firing shells with initial speed V_0 , or a lawn sprinkler is ejecting water at initial speed V_0, you are safe as long as you are outside the paraboloid of safety. Figure VII.1 shows trajectories for α = 20, 40, 60, 80, 100, 120, 140 and 160 degrees, and, as a dashed line, the paraboloid of safety. Notice how the range changes with α and that it is greatest for $\alpha = 45^\circ$.

FIGURE VII.1



Problem.

A gun projects a shell, in the absence of air resistance, at an initial angle α to the horizontal. The speed of projection varies with angle of projection and is given by

$$\text{Initial speed} = V_0 \cos \tfrac{1}{2}\alpha.$$

Show that, in order to achieve the greatest range on the horizontal plane, the shell should be projected at an angle to the horizontal whose cosine c is given by the solution of the equation

$$3c^3 + 2c^2 - 2c - 1 = 0.$$

Find the optimum angle to a precision of one arcminute.

7.2 *Air resistance proportional to the speed.*

As in the previous section, I shall write the x-component of the equation of motion, and of the first and second time integrals, in the left hand column, and the y-component in the right-hand column. The x-component of the air resistance per unit mass is $\gamma\dot{x}$ and the y-component is $\gamma\dot{y}$. Here γ is the damping constant, defined in Chapter 6, section 3. The x- and y-components of the initial velocity are, respectively, $V_0 \cos \alpha$ and $V_0 \sin \alpha$. It should be readily seen that the equations of motion and their time integrals are as follows:

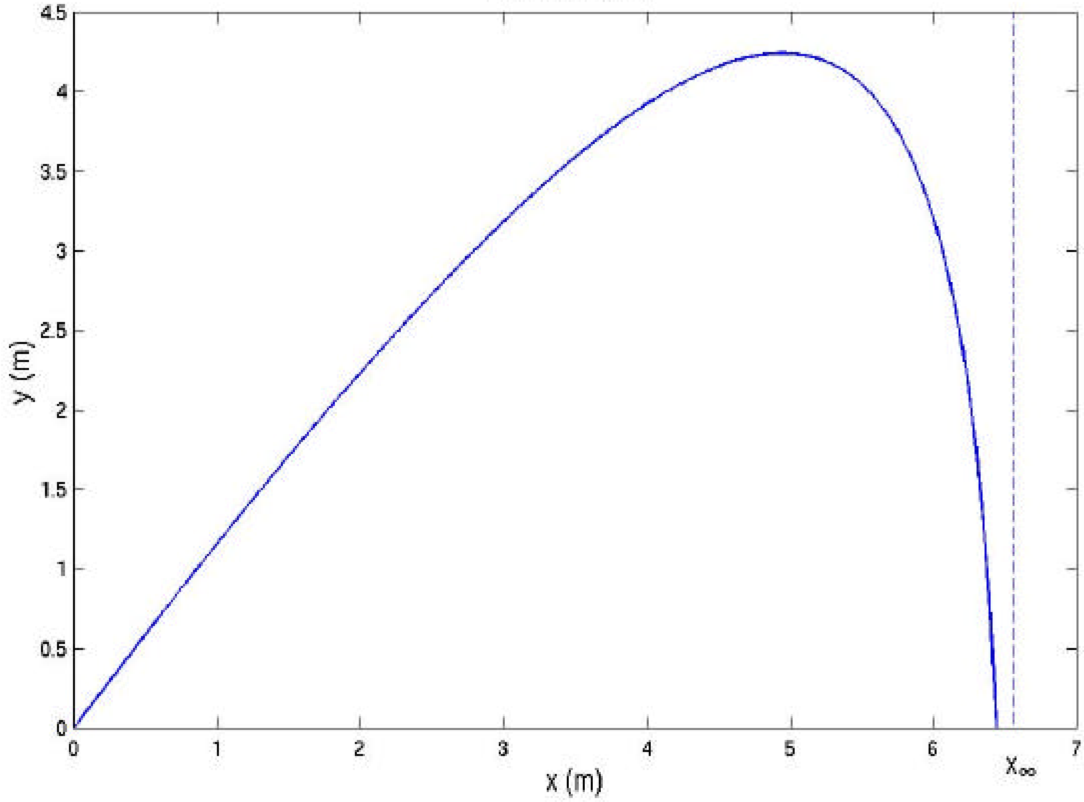
Horizontal	Vertical	
$\ddot{x} = -\gamma\dot{x}$	$\ddot{y} = -g - \gamma\dot{y}$	7.2.1*a,b*
$\dot{x} = u = V_0 \cos\alpha . e^{-\gamma t}$	$\dot{y} = v = V_0 \sin\alpha . e^{-\gamma t} - \hat{v}\left(1 - e^{-\gamma t}\right)$ where $\hat{v} = g/\gamma$	7.2.2*a,b*
$x = x_\infty\left(1 - e^{-\gamma t}\right)$ where $x_\infty = \dfrac{V_0 \cos\alpha}{\gamma}$	$y = \dfrac{1}{\gamma}\left(V_0 \sin\alpha + \hat{v}\right)\left(1 - e^{-\gamma t}\right) - \hat{v}t$	7.2.3*a,b*

(In case it is not "readily seen", for the horizontal motion refer to Chapter 6, section 3, especially equations 6.3.2, 6.3.3 and 6.3.5, and for the vertical motion refer to Chapter 6, section 3b, especially equations 6.3.24, 6.3.25 and 6.3.27.) It will be seen that, as $t \to \infty$, $u \to 0$, $v \to -\hat{v}$, $x \to x_\infty$. The *xy-equation to the trajectory* is the t-eliminant of equations 6.2.3*a* and 6.2.3*b*. After a small amount of algebra this is found to be:

$$y = \frac{x(V_0 \sin \alpha + \hat{v})}{V_0 \cos \alpha} + \frac{\hat{v}}{\gamma} \ln\left(1 - \frac{x}{x_\infty}\right) \qquad 7.2.4$$

This is illustrated in figure VII.2.

FIGURE VII.2



The *range on a horizontal plane* is found by setting x = 0, to obtain either

$$x = -A \ln(1 - x / x_\infty) \qquad 7.2.5$$

or

$$x = x_\infty \left(1 - e^{-x/A}\right), \qquad 7.2.6$$

where

$$A = \frac{\hat{v} V_0 \cos \alpha}{\gamma(V_0 \sin \alpha + \hat{v})}.$$

Example. Suppose $V_0 = 20 \text{ m s}^{-1}$

$\alpha = 50^{\circ}$

$g = 9.8 \text{ m s}^{-2}$

$\gamma = 1.96 \text{ s}^{-1} \quad (\therefore \hat{v} = 5 \text{ m s}^{-1})$

Then $A = 1.613\,870\,65$ m

and $x_{\infty} = 6.559\,057\,24$ m.

Try to find the range on the horizontal plane, using either equation 7.2.5 or 7.2.6, to nine significant figures. Which equation works best? Newton-Raphson may fail with a stupid first guess - but it should not be difficult to make a fairly intelligent first guess. I should not tell you, but figure VII.2 was calculated using the data of this example.

7.3 *Air resistance proportional to the square of the speed.*

Notation: $\mathbf{V}$ is the velocity, V is the speed. The horizontal and vertical components of the velocity are, respectively, $u = \dot{x} = V \cos \psi$ and $v = \dot{y} = V \sin \psi$. Here ψ is the angle that the instantaneous velocity $\mathbf{V}$ makes with the horizontal. The resistive force per unit mass is kV^2. The horizontal and vertical components of the resistive force per unit mass are $kV^2 \cos \psi$ and $kV^2 \sin \psi$ respectively. The launch speed is V_0 and the launch angle (i.e. the initial value of ψ) is α. Distance travelled from the launch point, measured along the trajectory, is s, and speed $V = \dot{s}$.
The equations of motion are:

Horizontal: $$\ddot{x} = -kV^2 \cos \psi \tag{7.3.1}$$

Vertical: $$\ddot{y} = -g - kV^2 \sin \psi. \tag{7.3.2}$$

These cannot be integrated as conveniently as in the previous cases, but we *can* get a simple relation between the horizontal component u of the speed and the intrinsic coordinate s. Thus, when we make use of $\ddot{x} = \dot{u}$, $V = \dot{s}$ and $V \cos \psi = u$, equation 7.3.1 takes the form

$$\dot{u} = -ku\dot{s}. \tag{7.3.3}$$

Integration, with initial condition $u = V_0 \cos \theta$, yields

$$u = V_0 \cos \alpha . e^{-ks}. \tag{7.3.4}$$

We can also obtain an exact explicit *intrinsic equation* to the trajectory by consideration of the *normal* equation of motion.

The *intrinsic equation* to any curve is a relation between the *intrinsic coordinates* (s , ψ). The rate at which the slope angle ψ changes as you move along the curve, i.e. $d\psi/ds$, is called the *curvature* at a point along the curve. If the slope is increasing with s, the curvature is positive. The reciprocal of the curvature at a point, $ds/d\psi$, is the *radius of curvature* at the point, denoted here by ρ.

The *normal* equation of motion is the equation $F = ma$ applied in a direction normal to the curve. The acceleration appropriate here is the *centripetal acceleration* V^2/ρ or $V^2 d\psi/ds$.

In a direction normal to the motion, the air resistance has no component, and gravity has a component $-g \cos \theta$. (It is *minus* because the curvature is clearly negative.) The normal equation of motion is therefore

$$V^2 \frac{d\psi}{ds} = -g\cos\theta. \qquad 7.3.5$$

But
$$V = \frac{u}{\cos\psi} = \frac{V_0 \cos\alpha . e^{-ks}}{\cos\psi} \qquad 7.3.6$$

Therefore
$$V_0^2 \cos^2\alpha . e^{-2ks} \frac{d\psi}{ds} = -g\cos^3\psi. \qquad 7.3.7$$

Separate the variables, and integrate, with appropriate initial conditions:

$$\int_\alpha^\psi \sec^3\psi d\psi = -\frac{g}{V_0^2 \cos^2\alpha}\int_0^s e^{2ks} ds. \qquad 7.3.8$$

From here it is good integration practice to show that the intrinsic equation is

$$\sec\psi\tan\psi - \sec\alpha\tan\alpha + \ln\left(\frac{\sec\psi + \tan\psi}{\sec\alpha + \tan\alpha}\right) = \frac{g}{kV_0^2\cos^2\alpha}\left(1 - e^{2ks}\right). \qquad 7.3.9$$

This equation is of the form

$$\sec\psi\tan\psi + \ln(\sec\psi + \tan\psi) = A - Be^{2ks}. \qquad 7.3.10$$

While it would be straightforward now to compute s as a function of ψ and hence to plot a graph of s versus ψ, we really want to show y as a function of x, and x and y as a function of time. I am indebted to Dario Bruni of Italy for the following analysis.

Let (x_1, y_1) be a point on the trajectory. When the projectile moves a short distance Δs, the new coordinates will be (x_2, y_2), where

$$x_2 = x_1 + \Delta s \cos \psi_1 \tag{7.3.11}$$

and

$$y_2 = y_1 + \Delta s \sin \psi_1, \tag{7.3.12}$$

provided that Δs is taken to be sufficiently small that the path between the two points is approximately a straight line. The calculation starts with $x_1 = y_1 = 0$ and $\psi = \alpha$. At each stage of the calculation, the new value of ψ can be calculated from equation 7.3.10. This can be done easily, for example, by Newton-Raphson iteration, since the derivative of the left hand side of the equation with respect to ψ is just $2\sec^3\psi$. Thus, with a sufficiently small interval Δs, the shape of the trajectory can be built up point by point.

While this gives us the shape of the trajectory, it tells us nothing about the time. To do this, we can write the equations of motion, equations 7.3.1 and 7.3.2 in the forms

$$\ddot{x} = -k\dot{x}\sqrt{\dot{x}^2 + \dot{y}^2} \tag{7.3.13}$$

and

$$\ddot{y} = -g - k\dot{y}\sqrt{\dot{x}^2 + \dot{y}^2}. \tag{7.3.14}$$

Let (x_1, y_1) be a point on the trajectory. After a short time Δt, the new coordinates will be (x_2, y_2), where

$$x_2 = x_1 + \dot{x}_1\Delta t + \tfrac{1}{2}\ddot{x}_1(\Delta t)^2 \tag{7.3.15}$$

and

$$y_2 = y_1 + \dot{y}_1\Delta t + \tfrac{1}{2}\ddot{y}_1(\Delta t)^2, \tag{7.3.16}$$

provided that Δt is taken to be sufficiently small that the acceleration between the two instants of time is approximately constant. Also, the new velocity components are given by

$$\dot{x}_2 = \dot{x}_1 + \ddot{x}_1\Delta t \tag{7.3.17}$$

and

$$\dot{y}_2 = \dot{y}_1 + \ddot{y}_1\Delta t. \tag{7.3.18}$$

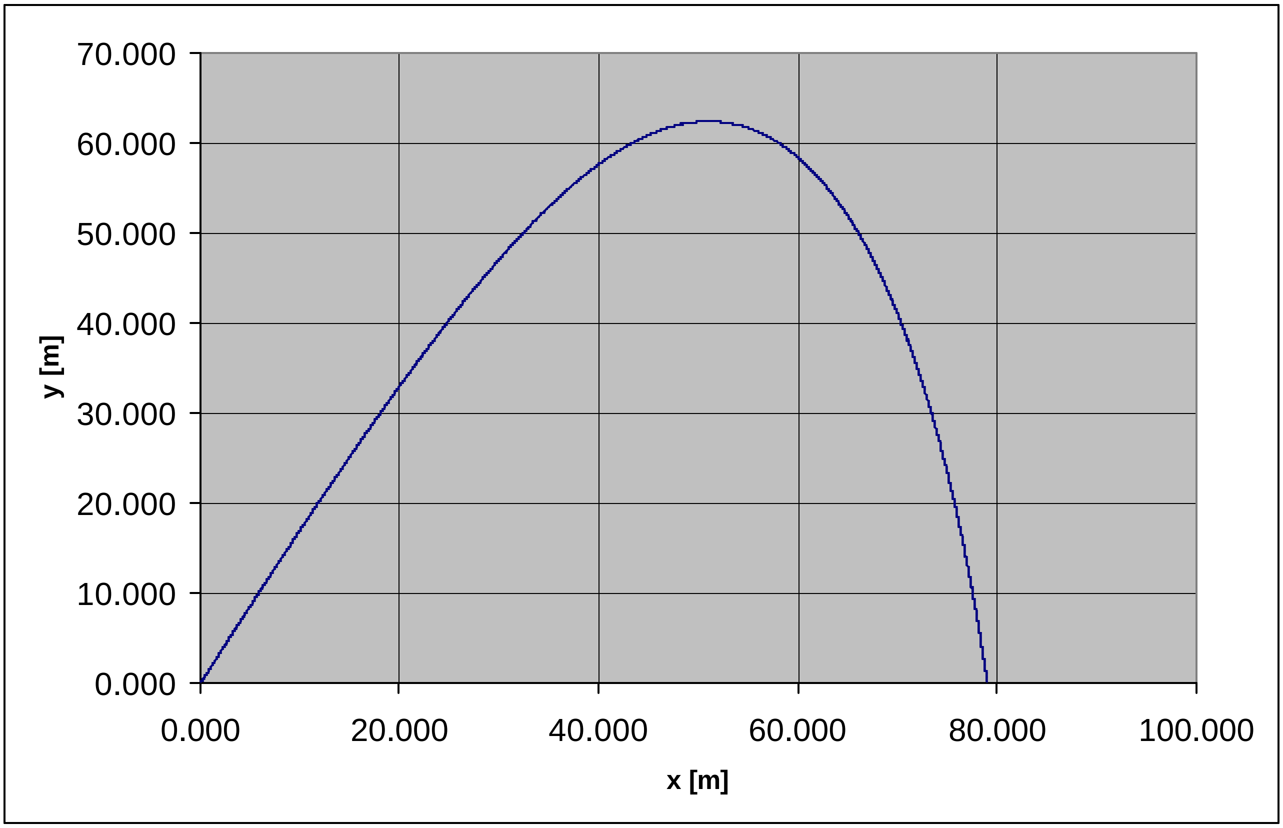
The calculation starts with

$$\dot{x} = V_0 \cos\alpha, \quad \dot{y} = V_0 \sin\alpha, \quad \ddot{x} = -kV_0^2 \cos\alpha, \quad \ddot{y} = -g - kV_0^2 \sin\alpha,$$

and after each increment Δt the new coordinates and velocity and acceleration components are calculated. The results of Sr Bruni's calculations are shown in figure VII.3 for

$$k = 0.0177 \text{ m}^{-1},\ V_0 = 90.5 \text{m s}^{-1},\ \alpha = 60^\circ,\ g = 9.8 \text{m s}^{-2}$$

FIGURE VII.3



Plotted with step by step method from intrinsic equation with Δs = 0.025 m.
Horizontal range 79.0 m; maximum height 62.4 m. Total flight duration 7.1 seconds.
The time taken to reach the maximum height is 2.8 seconds, so the descent time is longer than the ascent time.

An alternative approach has been given by Ambrose Okune, of Uganda. In Okune's analysis, he obtains explicit expressions for t, x and y in terms of the angle ψ. (In equation 7.3.10 we already have a relation between s and ψ.)

We start with equation 7.3.1, the horizontal equation of motion

$$\ddot{x} = -kV^2 \cos\psi = -kVV\cos\psi. \qquad 7.3.19$$

Now $\ddot{x} = \dot{u}$, $V = \sqrt{u^2 + v^2}$, and $V\cos\psi = u$, so that

$$\dot{u} = -ku\sqrt{u^2 + v^2}. \qquad 7.3.20$$

Similarly, equation 7.3.2, the vertical equation of motion, is

$$\ddot{y} = -g - kV^2 \sin\psi = -g - kVV\sin\psi, \qquad 7.3.21$$

and, with $\ddot{y} = \dot{v}$, $V = \sqrt{u^2 + v^2}$ and $V\sin\psi = v$, this becomes

$$\dot{v} = -g - kv\sqrt{u^2 + v^2}. \qquad 7.3.22$$

Now
$$\frac{\dot{v}}{\dot{u}} = \frac{dv}{du} = \frac{v}{u} + \frac{g}{ku\sqrt{u^2 + v^2}}. \qquad 7.3.23$$

Also $v = u\tan\psi$, so that

$$\frac{dv}{du} = \tan\psi + u\sec^2\psi\frac{d\psi}{du}. \qquad 7.3.24$$

On comparison of equations 7.3.23 and 7.3.24, we see that

$$\frac{g}{ku\sqrt{u^2 + v^2}} = u\sec^2\psi\, d\psi. \qquad 7.3.25$$

Upon substitution of $v = u\tan\psi$, this becomes

$$\frac{g}{ku^3} = \sec^3\psi\frac{d\psi}{du}\cdot \qquad 7.3.26$$

and hence
$$\frac{g}{k}\int u^{-3}du = \int \sec^3\psi\, d\psi. \qquad 7.3.27$$

Upon integration, we obtain

$$\frac{g}{ku^2} + \ln(\sec\psi + \tan\psi) + \sec\psi\tan\psi = A = \frac{g}{ku_0^2} + \ln(\sec\alpha + \tan\alpha) + \sec\alpha\tan\alpha. \qquad 7.3.28$$

From this, we obtain

$$u = \sqrt{\frac{g}{k}}\frac{1}{\sqrt{A - \ln(\sec\psi + \tan\psi) - \sec\psi\tan\psi}}, \qquad 7.3.29$$

and hence

$$v = \sqrt{\frac{g}{k}}\,\frac{\tan\psi}{\sqrt{A - \ln(\sec\psi + \tan\psi) - \sec\psi\tan\psi}}\,. \qquad 7.3.30$$

Thus we now have the velocity components explicitly in terms of the angle ψ.

For simplicity, let us write

$$\lambda = A - \ln(\sec\psi + \tan\psi) - \sec\psi\tan\psi. \qquad 7.3.31$$

Then the equations for the velocity components are

$$u = \sqrt{\frac{g}{k}}\,\frac{1}{\sqrt{\lambda}} \qquad 7.3.32$$

and

$$v = \sqrt{\frac{g}{k}}\,\frac{\tan\psi}{\sqrt{\lambda}}\,. \qquad 7.3.33$$

In the limit, as $u \to 0$, $\psi \to -90^\circ$, $y \to -\infty$, the motion approaches a vertical asymptote. As $\psi \to -90^\circ$, $\lambda \to -\sec\psi\tan\psi$, and hence $\operatorname{Lim}_{\psi\to-90^\circ}\frac{\tan\psi}{\sqrt{\lambda}} = -1$. Thus the limiting value of the vertical component of the velocity is $-\sqrt{\frac{g}{k}}$. This agrees precisely with what one would expect for a body falling vertically at terminal speed, with resistance proportional to the square of the speed (see equation 6.4.5).

We now aim to find an expression relating ψ to t, which we do by noting that

$$\frac{d\psi}{dt} = \frac{\frac{du}{dt}}{\frac{du}{d\psi}} = \frac{\frac{du}{dt}}{\frac{du}{d\lambda}\frac{d\lambda}{d\psi}}. \qquad 7.3.34$$

The derivative du/dt can be found from the horizontal equation of motion $\ddot{x} = -kV^2\cos\psi$, which can be written (because $u = V\cos\psi$ and $\ddot{x} = \dot{u}$) as $\dot{u} = -ku^2\sec\psi$. Then, making use of equation 7.3.32, we obtain

$$\frac{du}{dt} = -\frac{g}{\lambda}\sec\psi. \qquad 7.3.35$$

The derivative $du/d\lambda$ can be found from equation 7.3.32 and is

$$\frac{du}{d\lambda} = -\frac{1}{2}\sqrt{\frac{g}{k}}\frac{1}{\lambda^{3/2}}. \qquad 7.3.36$$

The derivative $d\lambda / d\psi$ can be found from equation 7.3.31 and is

$$\frac{d\lambda}{d\psi} = -2\sec^3\psi. \qquad 7.3.37$$

Thus the relation we seek is

$$\frac{d\psi}{dt} = -\sqrt{gk}\sqrt{\lambda}\cos^2\psi. \qquad 7.3.38$$

If the initial motion of the projectile at time zero makes an angle α with the horizontal, then integration of equation 7.3.38 gives the following expression for the subsequent time t when the motion makes an angle ψ with the horizontal.

$$t = \frac{1}{\sqrt{gk}}\int_\psi^\alpha \frac{d\psi}{\sqrt{\lambda}\cos^2\psi}. \qquad 7.3.39$$

Also $u = \frac{dx}{dt} = \frac{dx}{d\psi}\frac{d\psi}{dt}$. With u and $\frac{d\psi}{dt}$ given respectively by equations 7.3.32 and 7.3.38, we obtain

$$\frac{dx}{d\psi} = -\frac{1}{k\lambda\cos^2\psi}, \qquad 7.3.40$$

from which we can calculate x as a function of ψ:

$$x = \frac{1}{k}\int_\psi^\alpha \frac{d\psi}{\lambda\cos^2\psi}. \qquad 7.3.41$$

Further, $v = \frac{dy}{dt} = \frac{dy}{d\psi}\frac{d\psi}{dt}$. With v and $\frac{d\psi}{dt}$ given respectively by equations 7.3.33 and 7.3.38, we obtain

$$\frac{dy}{d\psi} = -\frac{\tan\psi}{k\lambda\cos^2\psi}, \qquad 7.3.42$$

from which we can calculate y as a function of ψ:

$$y = \frac{1}{k}\int_{\psi}^{\alpha} \frac{\tan\psi\, d\psi}{\lambda\cos^2\psi}. \qquad 7.3.43$$

Equations 7.3.39, 7.3.41 and 7.3.43 enable us to calculate t, x and y as a function of ψ, and hence to calculate any one of them in terms of any of the others. In each case a numerical integration is required, such as by Simpson's rule or by Gaussian quadrature, or other integration algorithm, and, as is always the case, sufficient points must be sampled to obtain adequate precision. Numerical integration of these equations, using the data of Dario Bruno's example above, produced the same $x : y$ trajectory as calculated for figure VII.3 by Bruno, and the $x : t$ and $y : t$ relations shown in figure VII.4.

I am greatly indebted to Dario Bruni and to Ambrose Okune for their interesting and instructive contributions to this section – an inspirational example of international scientific cooperation between, Italy, Uganda and Canada!

FIGURE VII.4