www.drshokuhi.com

CHAPTER 11

SIMPLE AND DAMPED OSCILLATORY MOTION

11.1 *Simple Harmonic Motion*

I am assuming that this is by no means the first occasion on which the reader has met simple harmonic motion, and hence in this section I merely summarize the familiar formulas without spending time on numerous elementary examples

Simple harmonic motion can be defined as follows: It a point P moves in a circle of radius a at constant angular speed ω (and hence period $2\pi/\omega$) its projection Q on a diameter moves with simple harmonic motion. This is illustrated in figure XI.1, in which the velocity and acceleration of P and of Q are shown as coloured arrows. The velocity of P is just $a\omega$ and its acceleration is the centripetal acceleration $a\omega^2$. As in Chapter 8 and elsewhere, I use blue arrows for velocity vectors and green for acceleration.

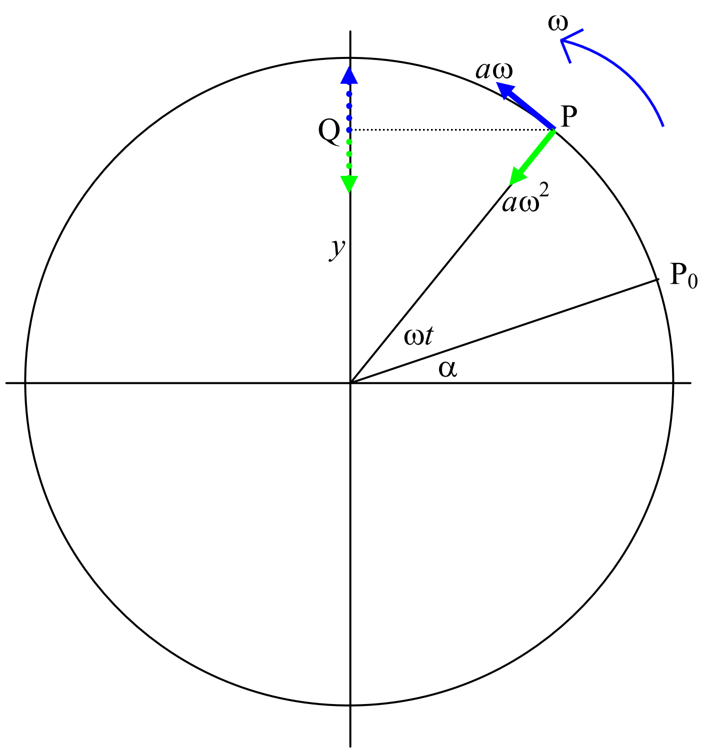


FIGURE XI.1

P_0 is the initial position of P - i.e. the position of P at time $t = 0$ - and α is the *initial phase angle.* At time t later, the phase angle is $\omega t + \alpha$. The projection of P upon a diameter is Q. The displacement of Q from the origin, and its velocity and acceleration, are

$$y = a\sin(\omega t + \alpha) \qquad 11.1.1$$

$$\upsilon = \dot{y} = a\omega\cos(\omega t + \alpha) \qquad 11.1.2$$

$$\ddot{y} = -a\omega^2 \sin(\omega t + \alpha). \qquad 11.1.3$$

Equations 11.1.2 and 11.1.3 can be obtained immediately either by inspection of figure XI.1 or by differentiation of equation 11.1.1. Elimination of the time from equations 11.1.1 and 11.1.2 and from equations 11.1.1 and 11.1.3 leads to

$$\upsilon = \dot{y} = \omega\left(a^2 - y^2\right)^{\frac{1}{2}} \qquad 11.1.4$$

and

$$\ddot{y} = -\omega^2 y. \qquad 11.1.5$$

An alternative definition of simple harmonic motion is to define as simple harmonic motion any motion that obeys the differential equation 11.1.5. We then have the problem of solving this differential equation. We can make no progress with this <u>unless we remember to write</u> $\ddot{y}$ as $\upsilon\frac{d\upsilon}{dy}$. (recall that we did this often in Chapter 6.) Equation 11.1.5 then immediately integrates to

$$\upsilon^2 = \omega^2\left(a^2 - y^2\right). \qquad 11.1.6$$

A further integration, with $\upsilon = dy/dt$, leads to

$$y = a\sin(\omega t + \alpha), \qquad 11.1.7$$

provided we remember to use the appropriate initial conditions. Differentiation with respect to time then leads to equation 11.1.2, and all the other equations follow.

Important Problem. Show that $y = a\sin(\omega t + \alpha)$ can be written

$$y = A\sin\omega t + B\cos\omega t, \qquad 11.1.8$$

where $A = a\cos\alpha$ and $B = a\sin\alpha$. The converse of these are

$a = \sqrt{A^2 + B^2}$, $\cos\alpha = \frac{A}{\sqrt{A^2+B^2}}$, $\sin\alpha = \frac{B}{\sqrt{A^2+B^2}}\cdot$ It is important to note that, if A and B are known, in order to calculate α without ambiguity of quadrant it is entirely necessary to calculate both cos α and sin α. It will not do, for example, to calculate α solely from $\alpha = \tan^{-1}(y/x)$, because this will give two possible solutions for α differing by 180°.

Show also that equation 11.1.8 can also be written

$$y = Me^{i\omega t} + Ne^{-i\omega t}, \qquad 11.1.9$$

where $M = \frac{1}{2}(B - iA)$ and $N = \frac{1}{2}(B + iA)$ and show that the right hand side of equation 11.1.9 is real.

The four large satellites of Jupiter furnish a beautiful demonstration of simple harmonic motion. Earth is almost in the plane of their orbits, so we see the motion of satellites projected on a diameter. They move to and fro in simple harmonic motion, each with different amplitude (radius of the orbit), period (and hence angular speed ω) and initial phase angle α.

11.2 *Mass Attached to an Elastic Spring.*

I am thinking of a mass *m* resting on a smooth horizontal table, rather than hanging downwards, because I want to avoid the unimportant distraction of the gravitational force (weight) acting on the mass. The mass is attached to one end of a spring of force constant *k*, the other end of the spring being fixed, and the motion is restricted to one dimension.

I suppose that the force required to stretch or compress the spring through a distance *x* is proportional to *x* and to no higher powers; that is, the spring obeys *Hooke's Law*. When the spring is stretched by an amount *x* there is a tension *kx* in the spring; when the spring is compressed by *x* there is a thrust *kx* in the spring. The constant *k* is the *force constant* of the spring.

When the spring is stretched by an distance *x*, its acceleration $\ddot{x}$ is given by

$$m\ddot{x} = -kx. \qquad 11.2.1$$

This is an equation of the type 11.1.5, with $\omega^2 = k/m$, and the motion is therefore simple harmonic motion of period

$$P = 2\pi/\omega = 2\pi\sqrt{\frac{m}{k}}\cdot \qquad 11.2.2$$

At this stage you should ask yourself two things: Does this expression have dimensions T? Physically, would you expect the oscillations to be slow for a heavy mass and a weak spring? The reader might be interested to know (and this is literally true) that when I first typed equation 11.2.2, I inadvertently typed $\sqrt{k/m}$, and I immediately spotted my mistake by automatically asking myself these two questions. The reader might also like to note that you can deduce that $P \propto \sqrt{m/k}$ by the method of dimensions, although you cannot deduce the proportionality constant 2π. Try it.

Energy Considerations. The work required to stretch (or compress) a Hooke's law spring by x is $\frac{1}{2}kx^2$, and this can be described as the potential energy or the elastic energy stored in the spring. I shall not pause to derive this result here. It is probably already known by the reader, or s/he can derive it by calculus. Failing that, just consider that, in stretching the spring, the force increases linearly from 0 to kx, so the average force used over the distance x is $\frac{1}{2}kx$ and so the work done is $\frac{1}{2}kx^2$.

If we assume that, while the mass is oscillating, no mechanical energy is dissipated as heat, the total energy of the system at any time is the sum of the elastic energy $\frac{1}{2}kx^2$ stored in the spring and the kinetic energy $\frac{1}{2}mv^2$ of the mass. (I am assuming that the mass of the spring is negligible compared with m.)

Thus
$$E = \tfrac{1}{2}kx^2 + \tfrac{1}{2}mv^2 \qquad 11.2.3$$

and there is a continual exchange of energy between elastic and kinetic. When the spring is fully extended, the kinetic energy is zero and the total energy is equal to the elastic energy then, $\frac{1}{2}ka^2$; when the spring is unstretched and uncompressed, the energy is entirely kinetic; the mass is then moving at its maximum speed $a\omega$ and the total energy is equal to the kinetic energy then, $\frac{1}{2}ma^2\omega^2$. Any of these expressions is equal to the total energy:

$$E = \tfrac{1}{2}kx^2 + \tfrac{1}{2}mv^2 = \tfrac{1}{2}ka^2 = \tfrac{1}{2}ma^2\omega^2. \qquad 11.2.4$$

11.3 *Torsion Pendulum*

A torsion pendulum consists of a mass of rotational inertia I hanging by a thin wire from a fixed point. If we assume that the torque required to twist the wire through an angle θ is proportional to θ and to no higher powers, then the ratio of the torque to the angle is called the torsion constant c. It depends on the shear modulus of the material of which the wire is made, is inversely proportional to its length, and, for a wire of circular cross-section, is proportional to the fourth power of its diameter. A thick wire is <u>much</u> harder to twist than a thin wire. Ribbonlike wires have comparatively small torsion constants. The work required to twist a wire through an angle θ is $\frac{1}{2}c\theta^2$.

When a torsion pendulum is oscillating, its equation of motion is

$$I\ddot{\theta} = -c\theta. \qquad 11.3.1$$

This is an equation of the form 11.1.5 and is therefore simple harmonic motion in which $\omega = \sqrt{c/I}$. This example, incidentally, shows that or second definition of simple harmonic motion (i.e. motion that obeys a differential equation of the form of equation 11.1.5) is a more general definition than our introductory description as the projection upon a diameter of uniform motion in a circle. In particular, do not imagine that ω here is the same thing as $\dot{\theta}$!

Exercise: Write down the torsional analogues of all the equations given for linear motion in sections 11.1 and 11.2.

11.4 *Ordinary Homogeneous Second-order Differential Equations.*

This is not a full mathematical course on differential equations, but it may be useful as a reminder for those who have already studied differential equations, and may even be just enough for our purposes for those who have not.

We suppose that $y = y(x)$ and y' denotes dy/dx. An ordinary homogenous second-order differential equation is an equation of the form

$$ay'' + by' + cy = 0, \qquad 11.4.1$$

and we have to find a function $y(x)$ which satisfies this. It turns out that it is quite easy to do this, although the nature of the solutions depends on whether b^2 is less than, equal to or greater than $4ac$.

A first point to notice is that, if $y = f(x)$ is a solution, so is $Af(x)$ − just try substituting this in the equation 11.4.1. If $y = g(x)$ is another solution, the same is true of g − i.e. $Bg(x)$ is also a solution. And you can also easily verify that any linear combination, such as

$$y = Af(x) + Bg(x), \qquad 11.4.2$$

is also a solution.

Now equation 11.4.1 is a second-order equation - i.e. the highest derivative is a second derivative - and therefore there can be only two arbitrary constants of integration in the solution - and we already have two in equation 11.4.1, and consequently there are no further solutions. All we have to do, then, is to find two functions that satisfy the differential equation.

It will not take long to discover that solutions of the form $y = e^{kx}$ satisfy the equation, because then $y' = ky$ and $y'' = k^2 y$, and, if you substitute these in equation 11.4.1, you obtain

$$\left(ak^2 + bk + c\right)y = 0. \qquad 11.4.3$$

You can always find two values of k that satisfy this, although these may be complex, which is why the nature of the solutions depends on whether b^2 is less than or greater than $4ac$. Thus the general solution is

$$y = Ae^{k_1 x} + Be^{k_2 x} \qquad 11.4.4$$

where k_1 and k_2 are the solutions of the equation

$$ax^2 + bx + c = 0. \qquad 11.4.5$$

There is one complication, however, if $b^2 = 4ac$, because then the two solutions of equation 11.4.5 are each equal to $-b/(2a)$. The solution of the differential equation is then

$$y = (A+B)\exp[-bx/(2a)] \qquad 11.4.6$$

and the two constants can be combined into a single constant $C = A + B$ so that equation 11.4.6 can be written

$$y = C\exp[-bx/(2a)]. \qquad 11.4.7$$

This solution has only one independent arbitrary constant, and so an additional solution must be possible. Let us try and see whether a function of the form

$$y = xe^{mx} \qquad 11.4.8$$

might be a solution of equation 11.4.1 From equation 11.4.8 we obtain $y' = (1+mx)e^{mx}$ and $y'' = m(2+mx)e^{mx}$. On substituting these into equation 11.4.1, remembering that $c = b^2/(4a)$, we obtain for the left hand side of equation 11.4.1, after some algebra,

$$\frac{e^{mx}}{4a}\left[(2am+b)^2 x + 4a(2am+b)\right]. \qquad 11.4.9$$

This is identically zero if $m = -b/(2a)$, and hence

$$y = x\exp[-bx/(2a)] \qquad 11.4.10$$

is a solution of equation 11.4.1. the general solution of equation 11.4.1, if $b^2 = 4ac$, is therefore

$$y = (C+Dx)\exp[-bx/(2a)]. \qquad 11.4.11$$

We shall discover what these solutions actually look like in the next section.

11.5 *Damped Oscillatory Motion*

As pointed out in section 11.2, the equation of motion for a mass m vibrating on the end of a spring of force constant k, in the absence of any damping, is

$$m\ddot{x} = -kx. \qquad 11.5.1$$

Here, I am assuming that the displacement x is a function of time, and a dot denotes d/dt.

However, in most real situations, there is some damping, or loss of mechanical energy, which is dissipated as heat. In the case of our example of a mass oscillating on a horizontal table, damping may be caused by friction between the mass and the table. For a mass hanging vertically from a spring, we might imagine the mass to be immersed in a viscous fluid. These are obvious examples. Slightly less obvious, it may be that the constant bending and stretching of the spring produces heat, and the motion is damped from this cause. In any case, in this analysis we shall assume that, in addition to the restoring force kx, there is also a damping force that is proportional to the speed at which the particle is moving. I shall denote the damping force by $b\dot{x}$. The equation of motion is then

$$m\ddot{x} = -kx - b\dot{x}. \qquad 11.5.2$$

If I divide by m and write ω_0^2 for k/m and γ for b/m, we obtain the equation of motion in its usual form

$$\ddot{x} + \gamma\dot{x} + \omega_0^2 x = 0. \qquad 11.5.3$$

Here γ is the *damping constant*, which we have already met in Chapter 10, and, from Section 11.4, we are ready to solve the differential equation 11.5.3. Indeed, we know that the general solution is

$$x = Ae^{k_1 t} + Be^{k_2 t}, \qquad 11.5.4$$

where k_1 and k_2 are the solutions of the quadratic equation

$$x^2 + \gamma x + \omega_0^2 = 0. \qquad 11.5.5$$

An exception occurs if $k_1 = k_2$, and we shall deal with that exceptional case in due course (subsection 11.5iii). Otherwise, k_1 and k_2 are given by

$$k_1 = -\tfrac{1}{2}\gamma + \sqrt{\tfrac{1}{4}\gamma^2 - \omega_0^2}\,, \quad k_2 = -\tfrac{1}{2}\gamma - \sqrt{\tfrac{1}{4}\gamma^2 - \omega_0^2}\,. \qquad 11.5.6$$

In section 11.5.3, we pointed out that the nature of the solution depends on whether b^2 is less than, equal to or greater than $4ac$, or, in the present case, upon whether γ is less than, equal to or greater than $2\omega_0$. These cases are referred to, respectively, as lightly damped, critically damped and heavily damped. We shall start by considering light damping.

11.5i *Light damping*: $\gamma < 2\omega_0$

Since $\gamma < 2\omega_0$, we have to write equations 11.5.6 as

$$k_1 = -\tfrac{1}{2}\gamma + i\sqrt{\omega_0^2 - \tfrac{1}{4}\gamma^2}\,, \quad k_2 = -\tfrac{1}{2}\gamma - i\sqrt{\omega_0^2 - \tfrac{1}{4}\gamma^2}\,. \qquad 11.5.7$$

Further, I shall write

$$\omega' = \sqrt{\omega_0^2 - \tfrac{1}{4}\gamma^2}\,. \qquad 11.5.8$$

Equation 11.5.4 is therefore

$$x = Ae^{-\frac{1}{2}\gamma t+i\omega' t} + Be^{-\frac{1}{2}\gamma t-i\omega' t} = e^{-\frac{1}{2}\gamma t}\left(Ae^{+i\omega' t} + Be^{-i\omega' t}\right). \qquad 11.5.9$$

If x is to be real, A and B must be complex. Also, since $e^{-i\omega' t} = \left(e^{i\omega' t}\right)^*$, B must equal A^*, where the asterisk denotes the complex conjugate.

Let $A = \frac{1}{2}(a - ib)$ and $B = \frac{1}{2}(a + ib)$, where a and b are real. Then the reader should be able to show that equation 11.5.9 can be written as

$$x = e^{-\frac{1}{2}\gamma t}\left(a\cos\omega' t + b\sin\omega' t\right). \qquad 11.5.10$$

And if $C = \sqrt{a^2 + b^2}$, $\sin\alpha = \frac{a}{\sqrt{a^2+b^2}}$, $\cos\alpha = \frac{b}{\sqrt{a^2+b^2}}$, the equation can be written

$$x = Ce^{-\frac{1}{2}\gamma t}\sin(\omega' t + \alpha). \qquad 11.5.11$$

Equations 11.5.10, 11 or 12 are three equivalent ways of writing the solution. Each has two arbitrary integration constants (A, B), (a, b) or (C, α), whose values depend on the *initial conditions* - i.e. on the values of x and $\dot{x}$ when $t = 0$. Equation 11.5.11 shows that the motion is a sinusoidal oscillation of period a little less than ω_0, with an exponentially decreasing amplitude.

To find C and α in terms of the initial conditions, differentiate equation 11.5.11 with respect to the time in order to obtain an equation showing the speed as a function of the time:

$$\dot{x} = Ce^{-\frac{1}{2}\gamma t}\left[\omega'\cos(\omega' t + \alpha) - \tfrac{1}{2}\gamma\sin(\omega' t + \alpha)\right]. \qquad 11.5.12$$

By putting $t = 0$ in equations 11.5.11 and 11.5.12 we obtain

$$x_0 = C\sin\alpha \qquad 11.5.13$$

and

$$(\dot{x})_0 = C\left(\omega'\cos\alpha - \tfrac{1}{2}\gamma\sin\alpha\right). \qquad 11.5.14$$

From these we easily obtain

$$\cot\alpha = \frac{1}{\omega'}\left[\frac{(\dot{x})_0}{x_0} + \frac{\gamma}{2}\right] \qquad 11.5.15$$

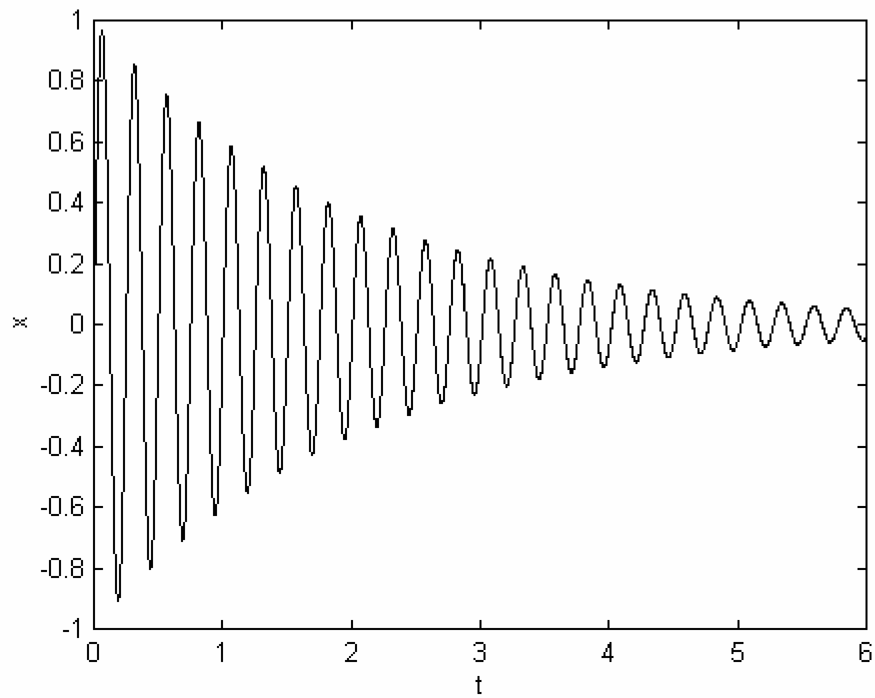
and

$$C = x_0\csc\alpha. \qquad 11.5.16$$

The quadrant of α can be determined from the signs of cot α and csc α, C always being positive. Note that the *amplitude* of the motion falls off with time as $e^{-\frac{1}{2}\gamma t}$, but the mechanical *energy*, which is proportional to the square of the amplitude, falls off as $e^{-\gamma t}$.

Figures XI.2 and XI.3 are drawn for $C = 1$, $\alpha = 0$, $\gamma = 1$. Figure XI.2 has $\omega_0 = 25\gamma$ and hence $x = e^{-\frac{1}{2}t}\sin 0.24.9949995t$, and figure XI.3 has $\omega_0 = 4\gamma$ and hence $x = e^{-\frac{1}{2}t}\sin 3.968626967t$.

FIGURE XI.2



Problem. Draw displacement : time graphs for an oscillator with $m = 0.02$ kg, $k = 0.08$ N m^{-1}, $\gamma = 1.5$ s^{-1}, $t = 0$ to 15 s, for the following initial conditions:

i. $x_0 = 0, \quad (\dot{x})_0 = 4 \text{ m s}^{-1}$

ii. $(\dot{x})_0 = 0, \quad x_0 = 3\text{m}$

iii. $(\dot{x}) = -2\text{m s}^{-1}, \quad x_0 = 2\text{m}$

Although the motion of a damped oscillator is not strictly "periodic", in that the motion does not repeat itself exactly, we could define a "period" $P = 2\pi/\omega'$ as the interval between two consecutive ascending zeroes. Extrema do not occur exactly halfway between consecutive zeroes, and the reader should have no difficulty is showing, by differentiation of equation 11.5.11, that extrema occur at times given by $\tan(\omega' t + \alpha) = 2\omega'/\gamma$. However, provided that the damping is not very large, consecutive extrema occur approximately at intervals of $P/2$. The ratio of consecutive maximum displacements is, then,

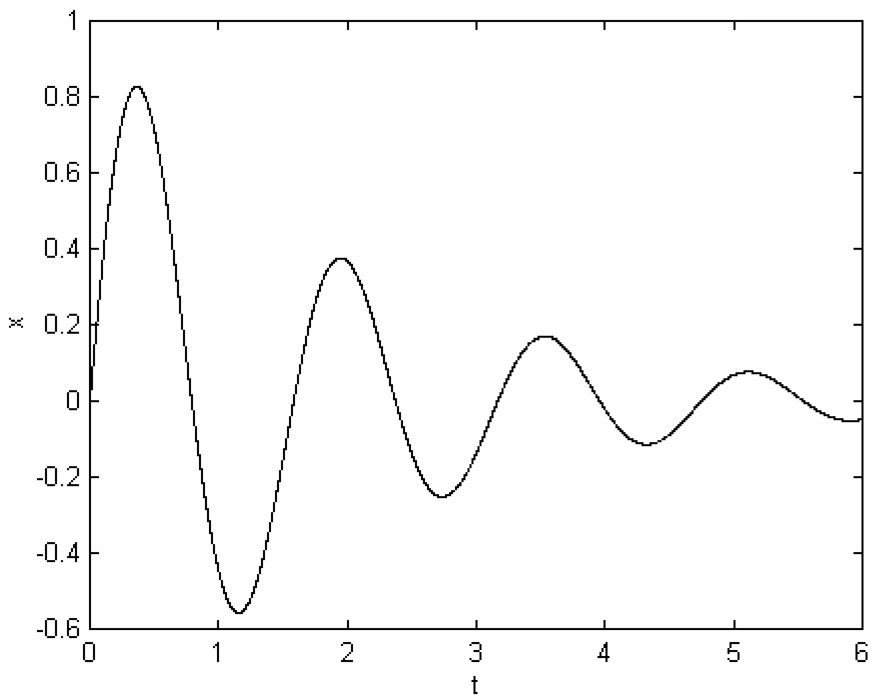
$$\frac{|\hat{x}_n|}{|\hat{x}_{n+1}|} = \frac{e^{-\frac{1}{2}\gamma t}}{e^{-\frac{1}{2}\gamma\left(t+\frac{1}{2}P\right)}}. \qquad 11.5.17$$

From this, we find that the *logarithmic decrement* is

$$\ln\left(\frac{|x_n|}{|x_{n+1}|}\right) = \frac{P\gamma}{4}, \qquad 11.5.18$$

from which the damping constant can be determined.

FIGURE XI.3



11.5ii *Heavy damping*: $\gamma > 2\omega_0$

The motion is given by equations 11.5.4 and 11.5.6 where, this time, k_1 and k_2 are each real and negative. For convenience, I am going to write $\lambda_1 = -k_1$ and $\lambda_2 = -k_2$. λ_1 and λ_2 are both real and positive, with $\lambda_2 > \lambda_1$ given by

$$\lambda_1 = \tfrac{1}{2}\gamma - \sqrt{\left(\tfrac{1}{2}\gamma\right)^2 - \omega_0^2}, \qquad \lambda_2 = \tfrac{1}{2}\gamma + \sqrt{\left(\tfrac{1}{2}\gamma\right)^2 - \omega_0^2}. \qquad 11.5.19$$

The general solution for the displacement as a function of time is

$$x = Ae^{-\lambda_1 t} + Be^{-\lambda_2 t}. \qquad 11.5.20$$

The speed is given by

$$\dot{x} = -A\lambda_1 e^{-\lambda_1 t} - B\lambda_2 e^{-\lambda_2 t}. \qquad 11.5.21$$

The constants *A* and *B* depend on the initial conditions. Thus:

$$x_0 = A + B \qquad 11.5.22$$

and

$$(\dot{x})_0 = -(A\lambda_1 + B\lambda_2). \qquad 11.5.23$$

From these, we obtain

$$A = \frac{(\dot{x})_0 + \lambda_2 x_0}{\lambda_2 - \lambda_1}, \qquad B = -\left[\frac{(\dot{x})_0 + \lambda_1 x_0}{\lambda_2 - \lambda_1}\right]. \qquad 11.5.24$$

Example 1. $x_0 \neq 0, \quad (\dot{x})_0 = 0.$

$$x = \frac{x_0}{\lambda_2 - \lambda_1}\left(\lambda_2 e^{-\lambda_1 t} - \lambda_1 e^{-\lambda_2 t}\right). \qquad 11.5.25$$

Figure XI.4 shows $x : t$ for $x_0 = 1$ m, $\lambda_1 = 1\ \text{s}^{-1}$, $\lambda_2 = 2\ \text{s}^{-1}$.

The displacement will fall to half of its initial value at a time given by putting $x/x_0 = 1/2$ in equation 11.5.25. In this will in general require a numerical solution. In our example, however, the equation reduces to $\frac{1}{2} = 2e^{-t} - e^{-2t}$, and if we let $u = e^{-t}$, this becomes $u^2 - 2u + \frac{1}{2} = 0$. The two solutions of this are $u = 1.707107$ or 0.292893. The first of these gives a negative *t*, so we want the second solution, which corresponds to $t = 1.228$ seconds.

The velocity as a function of time is given by

$$\dot{x} = -\frac{\lambda_1\lambda_2 x_0}{\lambda_2 - \lambda_1}\left(e^{-\lambda_1 t} - e^{-\lambda_2 t}\right). \qquad 11.5.26$$

This is always negative. In figure XI.5, is shown the speed, which is $|\dot{x}|$, or $-\dot{x}$, as a function of time, for our numerical example. Those who enjoy differentiating can show that the maximum speed is reached in a time $\frac{\ln(\lambda_2/\lambda_1)}{\lambda_2-\lambda_1}$, and that the maximum speed is $\frac{\lambda_1\lambda_2 x_0}{\lambda_2-\lambda_1}\left[\left(\frac{\lambda_1}{\lambda_2}\right)^{\frac{\lambda_2}{\lambda_2-\lambda_1}}-\left(\frac{\lambda_1}{\lambda_2}\right)^{\frac{\lambda_1}{\lambda_2-\lambda_1}}\right]$. (Are these dimensionally correct?) In our example, the maximum speed, reached at $t = \ln 2 = 0.6931$ seconds, is 0.5 m s^{-1}.

FIGURE XI.4

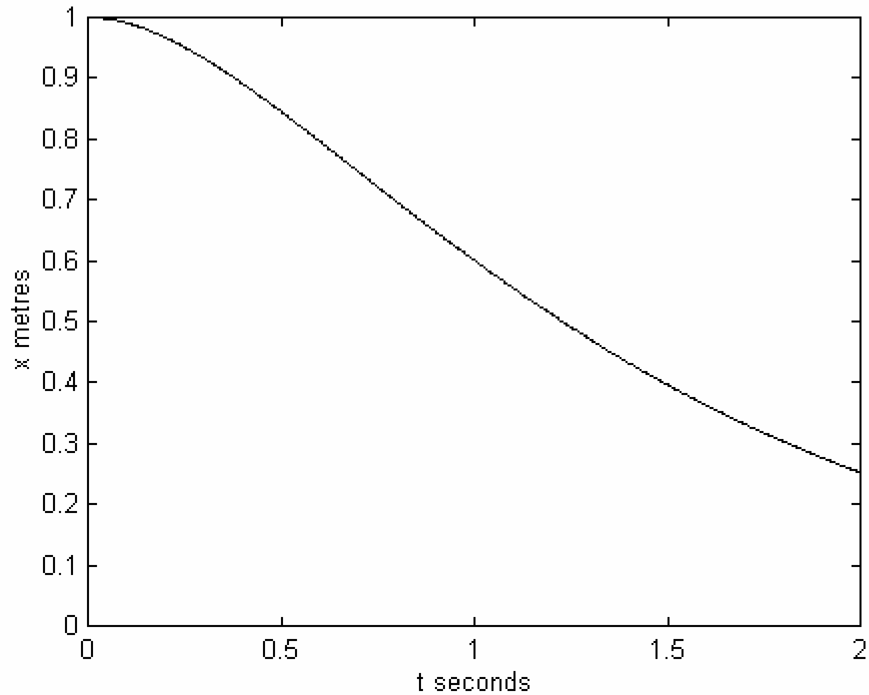
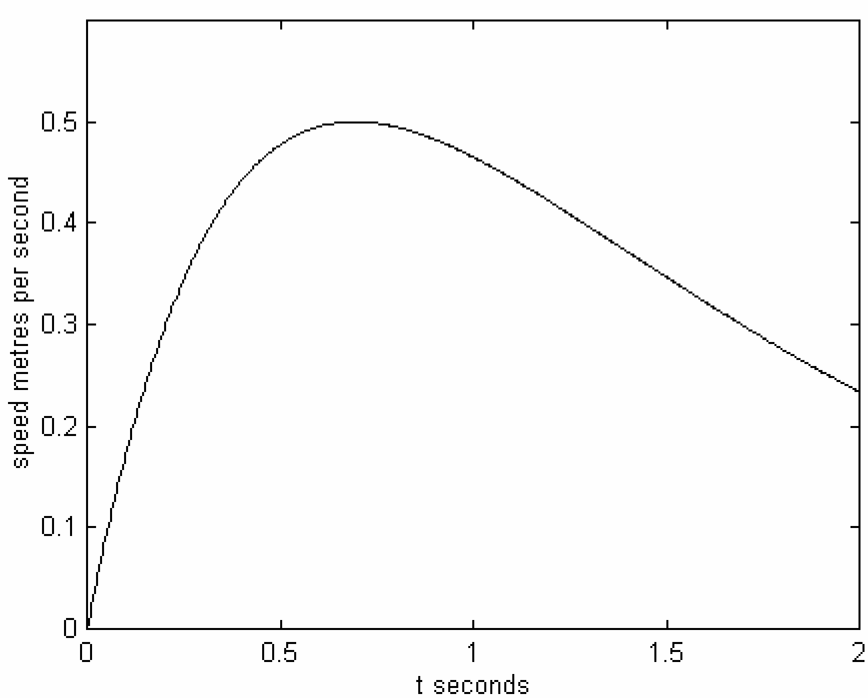


FIGURE XI.5



Example 2. $x_0 = 0, \quad (\dot{x})_0 = V (> 0)$.

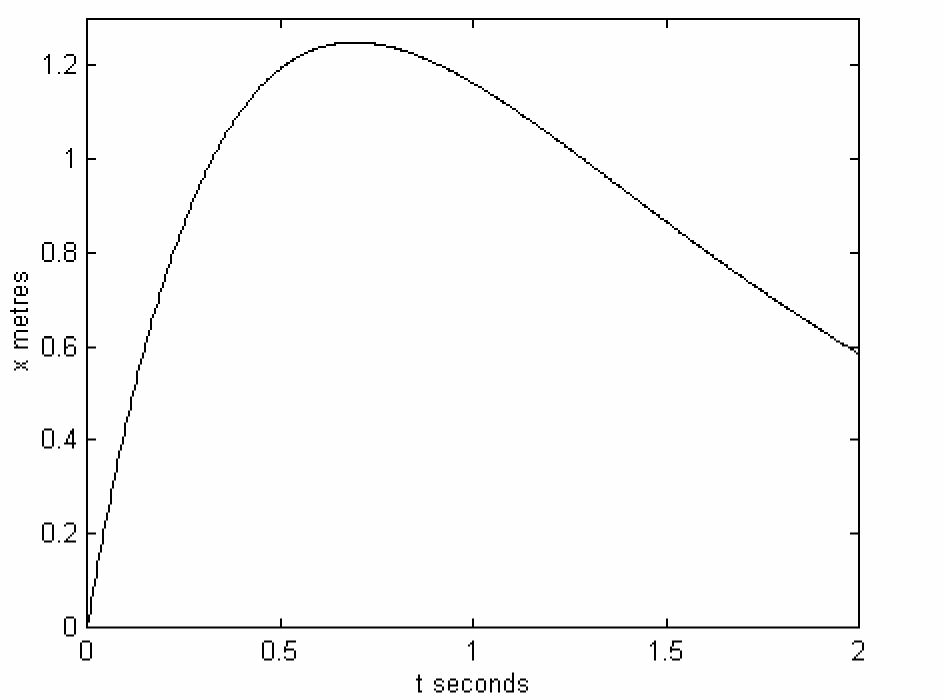
In this case it is easy to show that

$$x = \frac{V}{\lambda_2 - \lambda_1}\left(e^{-\lambda_1 t} - e^{-\lambda_2 t}\right). \qquad 11.5.26$$

It is left as an exercise to show that x reaches a maximum value of $\frac{V}{\lambda_2}\left(\frac{\lambda_1}{\lambda_2}\right)^{\frac{\lambda_1}{\lambda_2 - \lambda_1}}$ when $t = \frac{\ln(\lambda_2/\lambda_1)}{\lambda_2 - \lambda_1}$. Figure XI.6 illustrates equation 11.5.26 for $\lambda_1 = 1\ \text{s}^{-1}$, $\lambda_2 = 2\ \text{s}^{-1}$, $V = 5\ \text{m s}^{-1}$. The maximum displacement of 1.25 m is reached when $t = \ln 2 = 0.6831$ s. It is also left as an exercise to show that equation 11.5.26 can be written

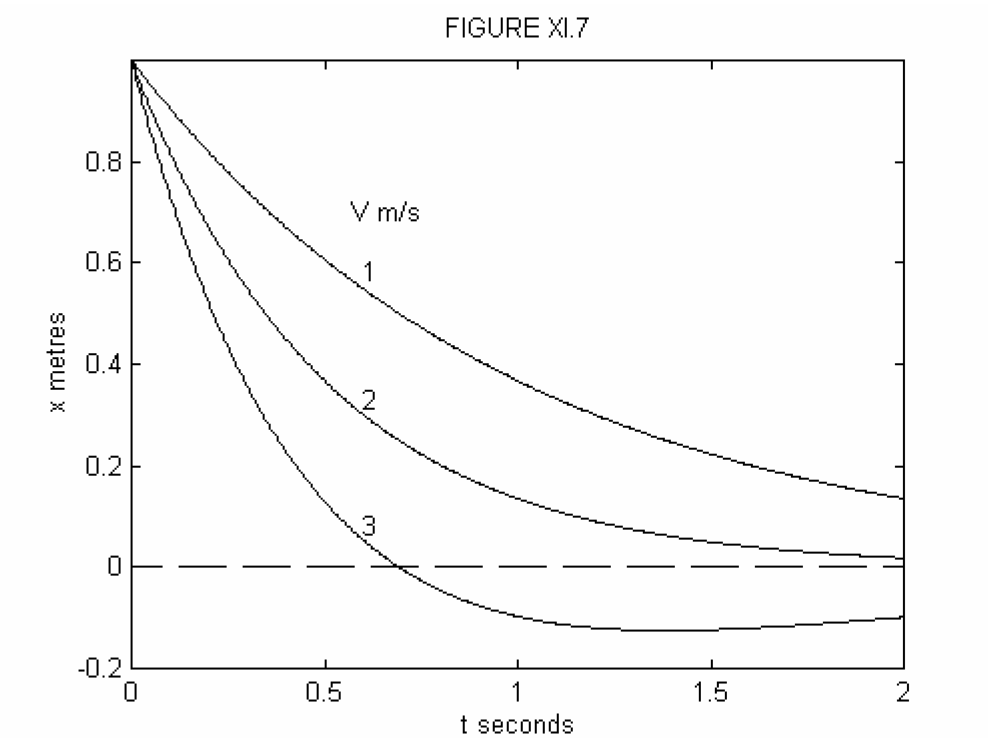
$$x = \frac{2Ve^{-\frac{1}{2}\gamma t}}{\lambda_2 - \lambda_1}\sinh\left(\tfrac{1}{4}\gamma^2 - \omega_0^2\right). \qquad 11.5.27$$

FIGURE XI.6



Example 3. $x_0 \neq 0, \quad (\dot{x})_0 = -V.$

This is the really exciting example, because the suspense-filled question is whether the particle will shoot past the origin at some finite time and then fall back to the origin; or whether it will merely tamely fall down asymptotically to the origin without ever crossing it. The tension will be almost unbearable as we find out. In fact, I cannot wait; I am going to plot x versus t in figure XI.7 for $\lambda_1 = 1\ \text{s}^{-1}$, $\lambda_2 = 2\ \text{s}^{-1}$, $x_0 = 1$ m, and three different values of V, namely 1, 2 and 3 m s^{-1}.



We see that if $V = 3$ m s^{-1} the particle overshoots the origin after about 0.7 seconds. If $V = 1$ m s^{-1}, it doesn't look as though it will ever reach the origin. And if $V = 2$ m s^{-1}, I'm not sure. Let's see what we can do. We can find out when it crosses the origin by putting $x = 0$ in equation 11.5.20, where A and B are found from equations 11.5.24 with $(\dot{x})_0 = -V$. This gives, for the time when it crosses the origin,

$$t = \frac{1}{\lambda_2 - \lambda_1} \ln\left(\frac{V - \lambda_1 x_0}{V - \lambda_2 x_0}\right). \qquad 11.5.28$$

Since $\lambda_2 > \lambda_1$, this implies that the particle will overshoot the origin if $V > \lambda_2 x_0$, and this in turn implies that, for a given V, it will overshoot only if

$$\gamma < \frac{V^2/x_0^2 + \omega_0^2}{V/x_0}. \qquad 11.5.29$$

For our example, $\lambda_2 x_0 = 2$ m s^{-1}, so that it just fails to overshoot the origin if $V = 2$ m s^{-1}. For $V =$ 3 m s^{-1}, it crosses the origin at $t = \ln 2 = 0.6931$ s. In order to find out how far past the origin it goes, and when, we can do this just as in *Example* 2. I make it that it reaches its maximum negative displacement of -0.125 m at $t = \ln 4 = 1.386$ s.

11.5iii *Critical damping*: $\gamma = 2\omega_0$

Before embarking on this section, you might just want to refresh your memory of differential equations as described in section 11.4.

In this case, λ_1 and λ_2 are each equal to $\gamma/2$. As discussed in section 4, the general solution is of the form

$$x = Ce^{-\frac{1}{2}\gamma t} + Dte^{-\frac{1}{2}\gamma t}, \qquad 11.5.30$$

which can also be written in the form

$$x = Ce^{-\frac{1}{2}\gamma t}(1 + at). \qquad 11.5.31$$

Either way, there are two arbitrary constants, which can be determined by the initial values of the displacement and speed. It is easy to show that

$$C = x_0 \text{ and } a = \frac{(\dot{x})_0}{x_0} + \tfrac{1}{2}\gamma. \qquad 11.5.32$$

The particle will not go through zero unless $V\left(= -(\dot{x})_0\right) > \tfrac{1}{2}\gamma x_0$. 11.5.33

I'll leave it to the reader to draw a graph of equation 11.5.19.

Ideally the hydraulic door closer that you see near the tops of doors in public buildings should be critically damped. This will cause the door to close fastest without slamming. And we have already used the physics of impulsive forces in problem 2.1 of Chapter 8 to work out where to place a door stop for minimum reaction on the hinges. Truly an understanding of physics is of enormous importance in achieving the task of closing a door!

A more subtle example is in the design of a moving-coil ammeter. In this instrument, the electric current is passed through a coil between the poles of a magnet, and the coil then swings around against the restoring force of a little spiral spring. The coil is wound on a light aluminium frame called a former, and, as the coil (and hence the former) moves in the magnetic field, a little current is induced in the former, and this damps the motion of the coil. In order that the coil and the pointer should move to the equilibrium position in the fastest possible time without oscillating, the system should be critically damped - which means that the rotational inertia and the electrical resistance of the little aluminium former has to be carefully designed to achieve this.

11.6 *Electrical Analogues*

A charged capacitor of capacitance C is connected in series with a switch and an inductor of inductance L. The switch is closed, and charge flows out of the capacitor and hence a current flows through the inductor. Thus while the electric field in the capacitor diminishes, the magnetic field in the inductor grows, and a back electromotive force (EMF) is induced in the inductor. Let Q be the charge in the capacitor at some time. The current I flowing from the positive plate is equal to $-\dot{Q}$. The potential difference across the capacitor is Q/C and the back EMF across the inductor is $L\dot{I} = -L\ddot{Q}$. The potential drop around the whole circuit is zero, so that $Q/C = -L\ddot{Q}$. The charge on the capacitor is therefore governed by the differential equation

$$\ddot{Q} = -\frac{Q}{LC}, \qquad 11.6.1$$

which is simple harmonic motion with $\omega_0 = 1/\sqrt{LC}$. You should verify that this has dimensions T^{-1}.

If there is a resistor of resistance R in the circuit, while a current flows through the resistor there is a potential drop $RI = -R\dot{Q}$ across it, and the differential equation governing the charge on the capacitor is then

$$LC\ddot{Q} + RC\dot{Q} + Q = 0. \qquad 11.6.2$$

This is damped oscillatory motion, the condition for critical damping being $R^2 = 4L/C$. In fact, it is not necessary actually to have a physical resistor in the circuit. Even if the capacitor and inductor were connected by superconducting wires of zero resistance, while the charge in the circuit is slopping around between the capacitor and the inductor, it will be radiating electromagnetic energy into space and hence losing energy. The effect is just as if a resistance were in the circuit.

If a battery of EMF E were in the circuit, the differential equation for Q would be

$$LC\ddot{Q} + RC\dot{Q} + Q = EC. \qquad 11.6.3$$

This is not quite an equation of the form 11.4.1, and I shan't spend time on it here. However, if we are interested in the current as a function of time, we just differentiate equation 11.6.3 with respect to time:

$$LC\ddot{I} + RC\dot{I} + I = 0. \qquad 11.6.4$$

The initial conditions are $I_0 = 0$ and $\left(\dot{I}\right)_0 = +E/L$, so the reader can easily write down all the solutions by comparison with section 11.5.