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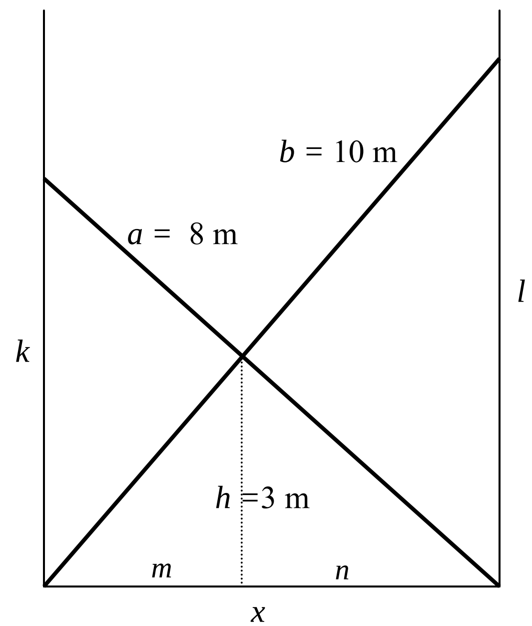
APPENDIX B
Solutions to Miscellaneous Problems

1.



By proportions, $\frac{h}{k} = \frac{n}{x}$ and $\frac{h}{l} = \frac{m}{x}$ and therefore $\frac{h}{k} + \frac{h}{l} = 1$.

Therefore by Pythagoras:

$$h\left(\frac{1}{\sqrt{a^2 - x^2}} + \frac{1}{\sqrt{b^2 - x^2}}\right) = 1.$$

Everything but x is known in this equation, which can therefore be solved for x. There are several ways of solving it; here's a suggestion. If we put in the numbers, the equation becomes

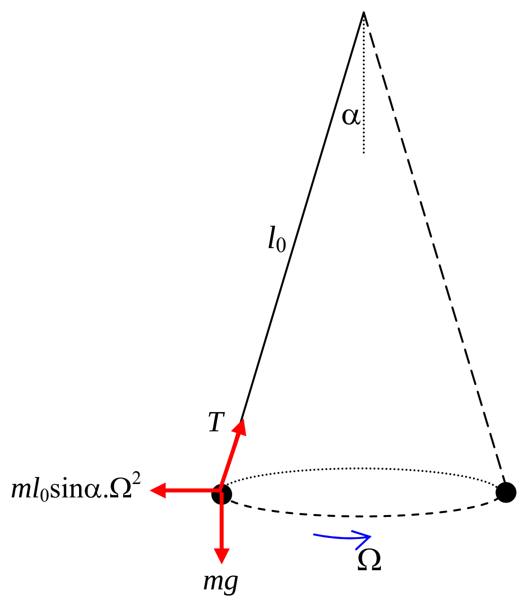
$$3\left(\frac{1}{\sqrt{64 - x^2}} + \frac{1}{\sqrt{100 - x^2}}\right) - 1 = 0.$$

Put $X = 100 - x^2$, and the equation becomes

$$3\left(\frac{1}{\sqrt{X - 36}} + \frac{1}{\sqrt{X}}\right) - 1 = 0.$$

This can be written $f(X) = 3(A+B) - 1 = 0$, where A and B are obvious functions of X. Differentiation with respect to X gives $f'(X) = -\frac{3}{2}(A^3 + B^3)$ and Newton-Raphson iteration $(X = X - f/f')$ soon gives X, from which it is then found that $x =$ 6.326 182 m.

2.



In the corotating frame the bob is in equilibrium under the action of three forces – its weight, the tension in the string and the centrifugal force. (If you don't like rotating reference frames and centrifugal force, it will be easy for you to do it "properly".) Resolve the forces perpendicular to the string: $ml_0 \sin\alpha.\Omega^2.\cos\alpha = mg\sin\alpha$, and the problem is finished.

3. (a) Raising or lowering the board doesn't apply any torques to the system, so the angular momentum L is conserved. That is,

$$L = ml^2 \sin^2\theta.\omega \quad \text{is constant.} \tag{1}$$

We also have that

$$g = l\cos\theta.\omega. \tag{2}$$

i. Eliminate ω from these equations. This gives:

$$l^3 \sin^3\theta \tan\theta = \frac{L^2}{gm^2}, \tag{3}$$

which is constant.

ii. Eliminate l from equations (1) and (2). This gives:

$$\omega^3 \cot^2 \theta = \frac{mg^2}{L}, \tag{4}$$

which is constant.

iii. Eliminate θ from equations (1) and (2). This gives:

$$\omega^3 \left(\omega l^2 - \frac{L}{m} \right) = g^2. \tag{5}$$

(Check the dimensions of all the equations.) Then we can get L/m from equation (1) and hence

$$\omega^3 \left(\omega l^2 - \Omega l_0^2 \sin^2 \alpha \right) = g^2,$$

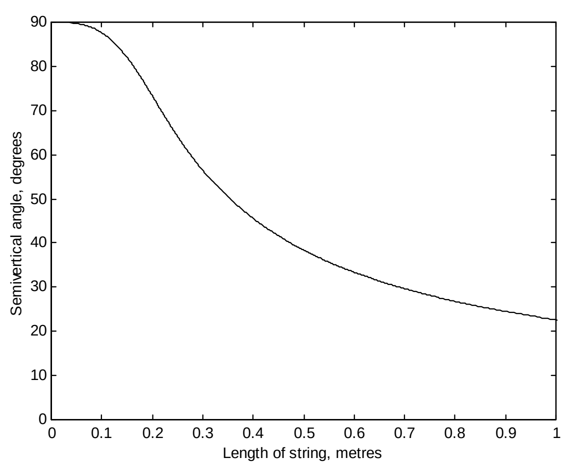
which is constant.

(b) i. $$l^3 \sin^3 \theta \tan \theta = l_0^3 \sin^3 \alpha \tan \alpha = 0.023675 \text{ m}^3.$$

Although we are asked to plot θ vertically versus l horizontally, it is easier, when working out numerical values, to calculate l as a function of θ. That is,

$$l = \frac{0.287142}{\sin \theta \sqrt[3]{\tan \theta}}.$$

(The number in the numerator is the cube root of 0.023675.)



For l = 40 cm = 0.4 m, the semivertical angle is given by

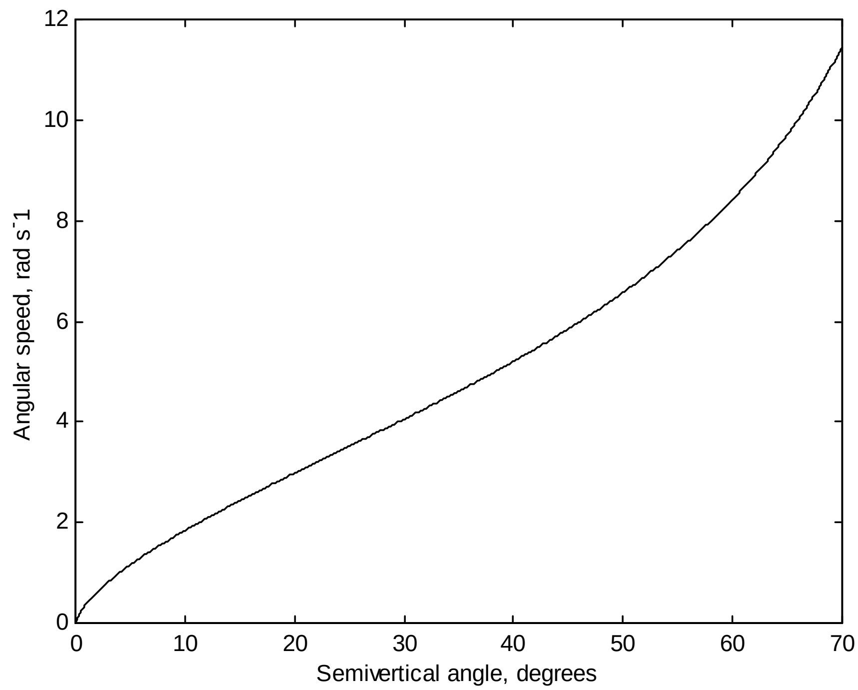
$$\sin^3\theta\tan\theta = 0.369923.$$

The solution to this is $\theta = 45^\circ\ 31'$.

(See section 1.4 of Celestial Mechanics if you need to know how to solve the equation $f(x) = 0$.)

(b) ii. $$\omega^3\cot^2\theta = \Omega^3\cot^2\alpha.$$

With the given data, this is $\omega^3 = 199.385\ \tan^2\theta$.



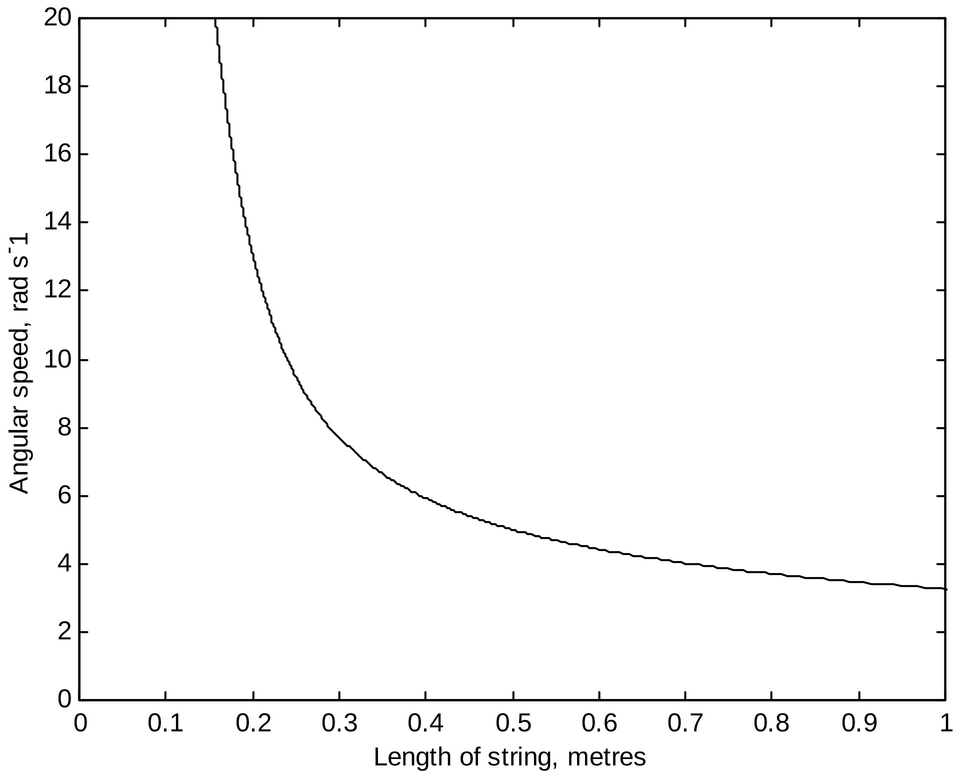
(b) iii. $$\omega^3\left(\omega l^2 - \Omega l_0^2\sin^2\alpha\right) = \Omega^3\left(\Omega l_0^2 - \Omega l_0^2\sin^2\alpha\right) = \Omega^4 l_0^2\cos^2\alpha.$$

That is, $\omega^3\left(\omega l^2 - a\right) = g^2$, where, with the given initial data,

$$a = 0.48168\ \text{m}^2\ \text{s}^{-1} \ \text{and}\ g^2 = 96.04\ \text{m}^2\ \text{s}^{-4}.$$

Although we are asked to plot ω vertically versus l horizontally, it is easier, when working out numerical values, to calculate l as a function of ω. That is,

$$l^2 = \frac{g^2}{\omega^4} + \frac{a}{\omega} \cdot$$

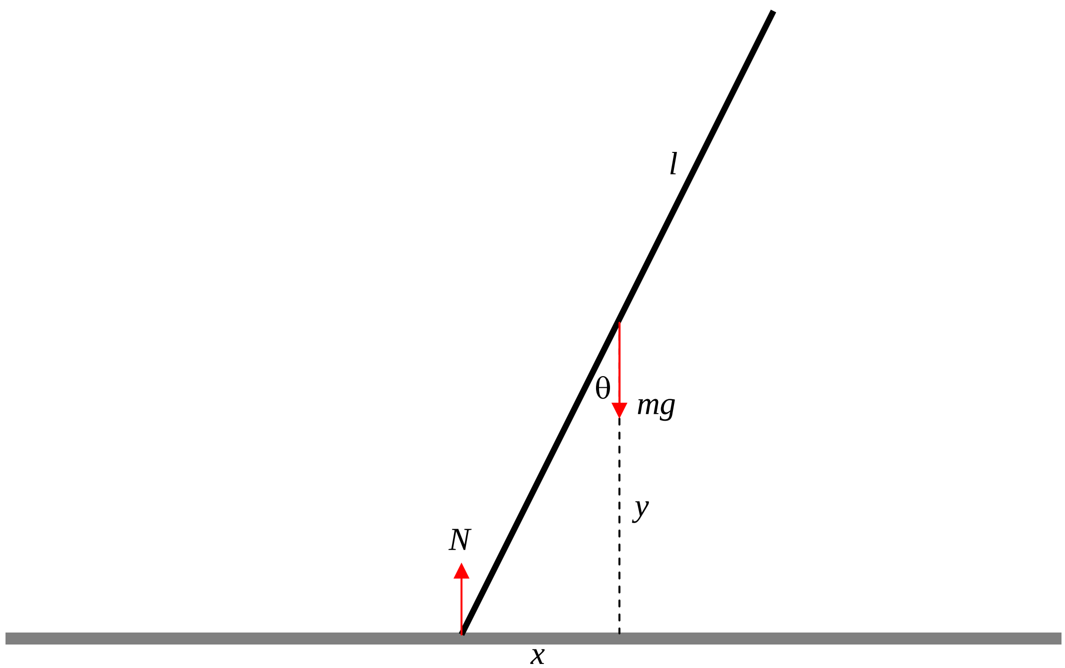


To solve the above equation for ω might be slightly easier with the substitution of u for 1/ω:

$$g^2u^4 + au - l^2 = 0.$$

With $l = 0.6$ m, this gives $u = 0.226121$ rad^{-1} s, and hence ω = 4.422 rad s^{-1}. As in part (b) i, it is necessary to know how to solve the equation $f(x) = 0$. See section 1.4 of Celestial Mechanics if you need to know how.

4. There are no horizontal forces, because the table is smooth. Therefore the centre of mass of the rod falls vertically.



From energy considerations

$$\tfrac{1}{2}m\dot{y}^2 + \tfrac{1}{2}\left(\tfrac{1}{3}ml^2\right)\dot{\theta}^2 + mgy = \text{constant}. \qquad (1)$$

But $y = l\cos\theta$ and therefore $\dot{y} = -l\sin\theta.\dot{\theta}$.

$\therefore$
$$(3\sin^2\theta + 1)l\,\dot{\theta}^2 + 6g\cos\theta = C. \qquad (2)$$

Initially $\theta = \dot{\theta} = 0$, $\therefore C = 6g$.

$\therefore$
$$\dot{\theta}^2 = \frac{6g(1-\cos\theta)}{l(3\sin^2\theta + 1)}. \qquad (3)$$

Also, since $\dot{y}^2 = l^2\sin^2\theta\,\dot{\theta}^2$ and $\dot{x}^2 = l^2\cos^2\theta\,\dot{\theta}^2$,

we obtain
$$\dot{y}^2 = \frac{6gl\sin^2\theta(1-\cos\theta)}{3\sin^2\theta + 1} \qquad (4)$$

and
$$\underline{\underline{\dot{x}^2 = \frac{6gl\cos^2\theta(1-\cos\theta)}{3\sin^2\theta+1}}}\,. \tag{5}$$

Of course $\dot\theta$ and $\dot y$ increase monotonically with θ; but $\dot x$ starts and finishes at zero, and must go through a maximum. With $c = \cos\theta$, equation (5) can be written

$$\dot{x}^2 = \frac{6glc(1-c)}{4-3c^2}\,, \tag{6}$$

and by differentiating $\dot{x}^2$ with respect to c, we see that $\dot{x}^2$ is greatest at an angle θ given by

$$3c^3 - 12c + 8 = 0, \tag{7}$$

the solution of which is $\underline{\underline{\theta = 37^\circ\ 50'}}$.

If the length of the rod is 1 m (l = 0.5 m) and $\dot x$ = 1 m s^{-1}, equation (6) becomes

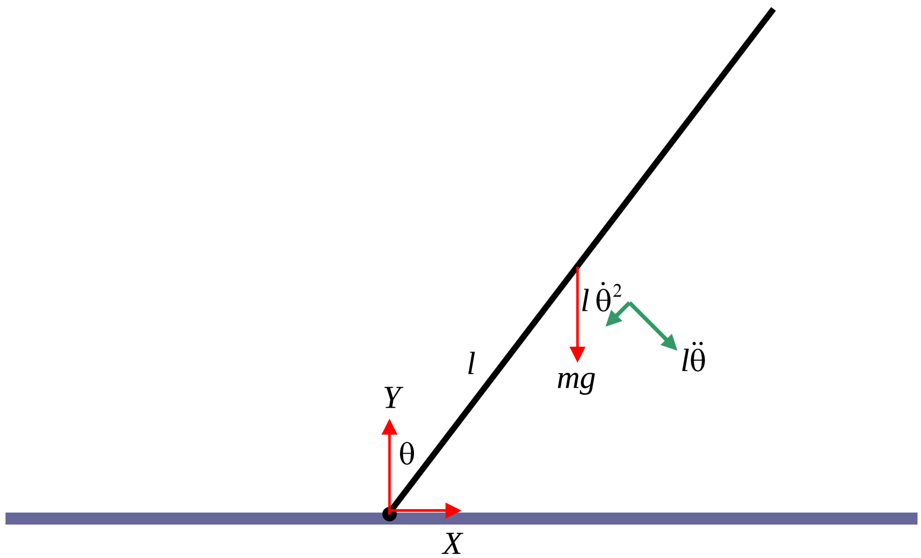
$$26.4c^2 - 29.4c + 4 = 0, \tag{8}$$

and the two solutions are $\underline{\underline{\theta = 17^\circ\ 15' \text{ and } 80^\circ\ 52'}}$.

The reader who has done all the problems so far will be aware of the importance of being able instantly to solve the equation $f(x) = 0$. If you have not already done so, you should write a computer or calculator program that enables you to do this instantly and at a moment's notice. See section 1.4 of Celestial Mechanics if you need to know how.

If you want to find the normal reaction N of the table on the lower end of the rod, you could maybe start with the vertical equation of motion $m\ddot y = N - mg$. Differentiate equation (4): $2\dot y \ddot y =$ whatever, and the use equation (4) again for $\dot y$. This looks like rather heavy and uninteresting algebra to me, so I shan't pursue it. There may be a better way...

5. In the figure below I have marked in red the forces on the rod, namely its weight mg and the horizontal and vertical components X and Y of the reaction of the hinge on the rod. I have also marked, in green, the transverse and radial components of the acceleration of the centre of mass. The transverse component is $l\ddot{\theta}$ and the radial component is the centripetal acceleration $l\dot{\theta}^2$.



From consideration of the moment of the force mg about the lower end of the rod, it is evident that the angular acceleration is

$$\ddot{\theta} = \frac{3g\sin\theta}{4l}, \tag{1}$$

and by writing $\ddot{\theta}$ as $\dot{\theta}\,d\dot{\theta}/d\theta$ and integrating (with initial conditions $\theta = \dot{\theta} = 0$), or from energy considerations, we obtain the angular speed:

$$\dot{\theta}^2 = \frac{3g(1-\cos\theta)}{2l}. \tag{2}$$

The horizontal and vertical equations of motion are:

$$X = ml(\ddot{\theta}\cos\theta - \dot{\theta}^2\sin\theta) \tag{3}$$

and

$$mg - Y = ml(\ddot{\theta}\sin\theta + \dot{\theta}^2\cos\theta). \tag{4}$$

(As ever, check the dimensions - and count the dots!)

After substitution for $\ddot{\theta}$ and $\dot{\theta}^2$ we find

$$X = \tfrac{3}{4} mg \sin\theta (3\cos\theta - 2) \tag{5}$$

and

$$Y = \tfrac{1}{4} mg (1 - 3\cos\theta)^2 . \tag{6}$$

The results follow immediately.

6. Call the length of the rod $2l$. Initially the height above the table of its centre of mass is $l \cos 40^\circ$, and its gravitational potential energy is $mg\, l \cos 40^\circ$. When it hits the table at angular speed ω, its kinetic energy is $\tfrac{1}{2} I\omega^2 = \tfrac{1}{2}\left(\tfrac{4}{3} ml^2\right)\omega^2 = \tfrac{2}{3} ml^2\omega^2$. Therefore,

$$\omega = \sqrt{\frac{3g\cos 40^\circ}{2l}} = \underline{\underline{4.746 \text{ rad s}^{-1} = 271.9 \text{ deg s}^{-1}}} .$$

To find the time taken, you can use equation 9.2.10:

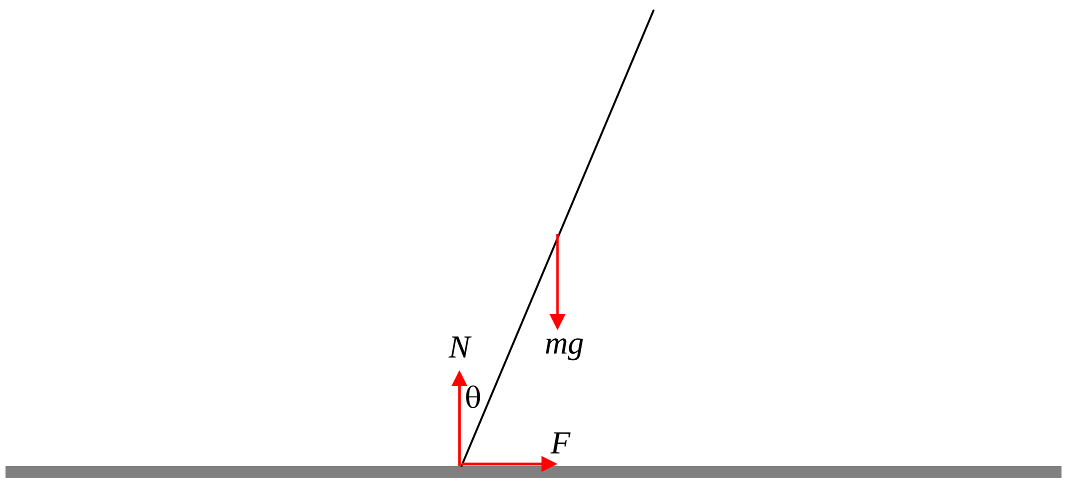
$$t = \sqrt{\frac{I}{2}} \int_{40^\circ}^{90^\circ} \frac{d\theta}{\sqrt{E - V(\theta)}} . \tag{7}$$

Here, $I = \tfrac{4}{3} ml^2$, $E = mgl \cos 40^\circ$, $V(\theta) = mgl\cos\theta$ and therefore

$$t = \sqrt{\frac{2l}{3g}} \int_{40}^{90} \frac{d\theta}{\sqrt{\cos 40^\circ - \cos\theta}} . \tag{8}$$

The magnitude of the quantity before the integral sign is 0.184428 s. To find the value of the integral requires either that you be an expert in elliptic integrals or (more likely and more useful) that you know how to integrate numerically (see Celestial Mechanics 1.2.) I make the value of the integral 2.187314, so that the time taken is $\underline{\underline{\text{0.4034 seconds}}}$. When integrating, note that the value of the integrand is infinite at the lower limit. How to deal with this difficulty is dealt with in Celestial Mechanics 1.2. It cannot be glossed over.

7. Here is the diagram. The forces are the weight mg of the rod, and the force of the table on the rod. However, I have resolved the latter into two components – the *normal* reaction N of the table on the rod, and the frictional force F, which may be either to the left or the right, depending on whether rod is tending to slip towards the right or the left. The magnitude of F is less than μN as long as the rod is not jus about the slip. When the rod is just about to slip, $F = \mu N$, μ being the coefficient of limiting static friction.



Just as in Problem 5, the equations of motion, as long as the rod does not slip, are

$$F = \tfrac{3}{4} mg \sin\theta (3\cos\theta - 2) \tag{1}$$

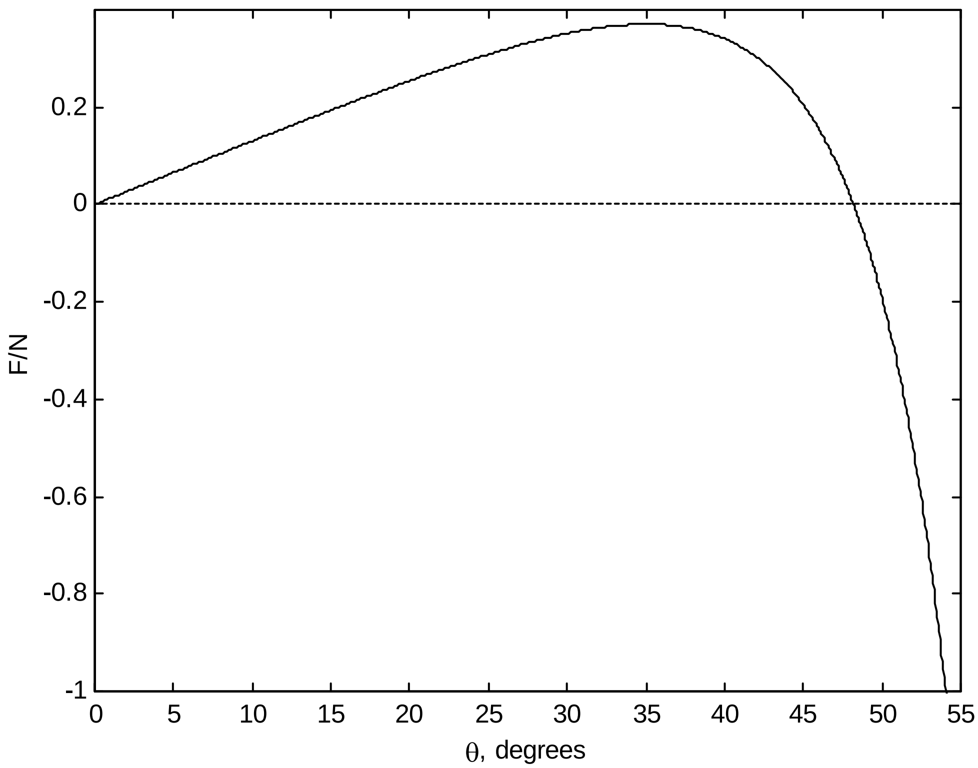
and

$$N = \tfrac{1}{4} mg (1 - 3\cos\theta)^2 . \tag{2}$$

$\therefore$

$$\frac{F}{N} = \frac{3\sin\theta(3\cos\theta - 2)}{(1 - 3\cos\theta)^2} . \tag{3}$$

The figure below shows F/N as a function of θ. One sees that, as the rod falls over, F/N increases, and, as soon as it attains a value of μ, the rod will slip. We see, however, that F/N reaches a maximum value, and by calculus we can determine that it reaches a maximum value of $15\sqrt{10}/128 = 0.3706$ when $\theta = \cos^{-1}(\tfrac{9}{11}) = 35^\circ\ 06'$. If $\mu <$ 0.3706, the bottom of the rod will slip before $\theta = 35^\circ\ 06'$. If, however, $\mu > 0.3706$, the rod will not have slipped by the time $\theta = 35^\circ\ 06'$, and it is safe for a while as F/N starts to decrease. When θ reaches $\cos^{-1}(\tfrac{2}{3}) = 48^\circ\ 11'$, the frictional force changes sign and thereafter acts to the left. (The frictional force of the table on the rod acts to the left; the frictional force of the rod on the table acts to the right.) We know by now (since the rod survived slipping before $\theta = 35^\circ\ 06'$, that the magnitude of F/N can be at least as large



as 0.3706, and it doesn't reach this until $\theta = 51^\circ\ 15'$. Therefore, if the rod hasn't slipped by $\theta = 35^\circ\ 06'$, it won't slip before $\theta = 51^\circ\ 15'$. But after that it is in danger again of slipping. F/N becomes infinite ($N = 0$) when $\theta = \cos^{-1}(\frac{1}{3}) = 70^\circ\ 32'$, so it will certainly slip (to the right) before then.

If $\mu = 0.25$, the rod will slip to the left when

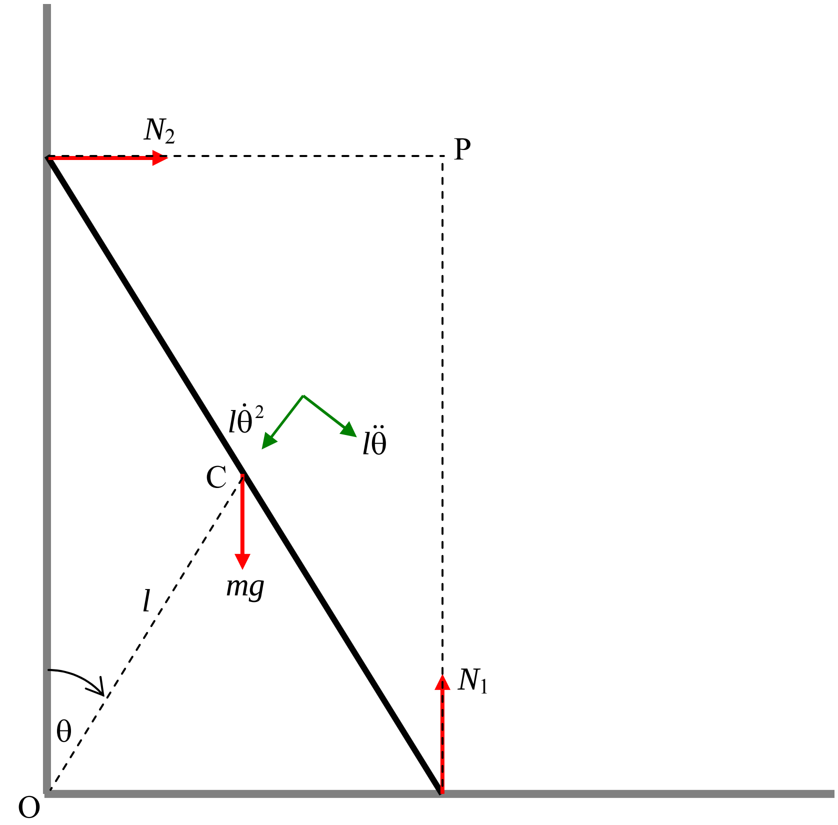
$$\frac{3\sin\theta(3\cos\theta - 2)}{(1-3\cos\theta)^2} = \frac{1}{4}, \text{ or } \theta = 19^\circ\ 39'.$$

If $\mu = 0.75$, the rod will slip to the right when

$$\frac{3\sin\theta(3\cos\theta - 2)}{(1-3\cos\theta)^2} = -\frac{3}{4}, \text{ or } \theta = 53^\circ\ 07'.$$

Again, it is very necessary that you prepare for yourself a program that will instantly solve the equation $f(x) = 0$.

8.



Let the length of the ladder be $2l$. By geometry, the distance OC remains equal to l throughout the motion; therefore C describes a circle of radius l, centre O. I have marked in, in green, the radial and transverse components of the acceleration of C, namely $l\dot{\theta}^2$ and $l\ddot{\theta}$. The angular speed of the ladder is $\dot{\theta}$ and the linear speed of the centre of mass C is $l\dot{\theta}$. I have also marked, in red the three forces acting on the ladder, namely its weight and the reactions of the floor and the wall on the ladder.

The angular speed $\dot{\theta}$ can be obtained from energy considerations. That is, the loss of potential energy in going from angle α to the vertical to angle θ is equal to the gain in translational and kinetic energies:

$$mgl(\cos\alpha - \cos\theta) = \tfrac{1}{2}m(l\dot{\theta})^2 + \tfrac{1}{2}(\tfrac{1}{3}ml^2)\dot{\theta}^2.$$

$$\therefore \qquad \dot{\theta}^2 = \frac{3g}{2l}(\cos\alpha - \cos\theta). \qquad (1)$$

The angular acceleration $\ddot{\theta}$ can be obtained by considering that the total moment of all forces about P is $mgl\sin\theta$, and the rotational inertia is $\tfrac{4}{3}ml^2$:

$$mgl\sin\theta = \tfrac{4}{3}ml^2\ddot{\theta}. \qquad (2)$$

The vertical and horizontal equations of motion are:

$$N_2 = m(l\ddot{\theta}\cos\theta - l\dot{\theta}^2\sin\theta) \quad (3)$$

and
$$mg - N_1 = m(l\ddot{\theta}\sin\theta + l\dot{\theta}^2\cos\theta), \quad (4)$$

although we need only the first of these, because we wish to find out when $N_2 = 0$.

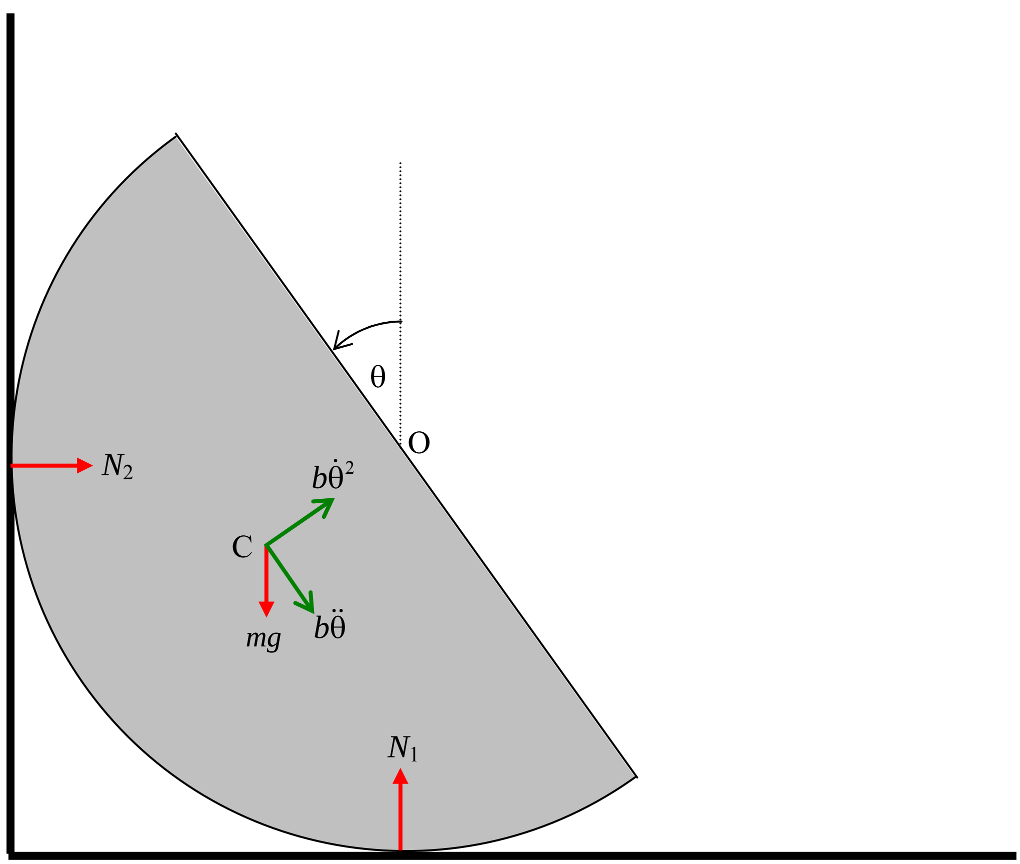
On substitution for $\ddot{\theta}$ and $\dot{\theta}^2$ we find that

$$N_2 = \tfrac{3}{4}mg\sin\theta(3\cos\theta - 2\cos\alpha) \quad (5)$$

and
$$N_1 = \tfrac{1}{4}mg(1 - 6\cos\alpha\cos\theta + 9\cos^2\theta). \quad (6)$$

We need only the first of these to see that N_2 becomes zero (and hence the upper end loses contact with the wall) $\cos\theta = \frac{2}{3}\cos\alpha$.

9.



It will, I think, be agreed that the point O remains fixed in space as long as the semicylinder remains in contact with wall and floor. Therefore the centre of mass C moves in a circle around O. We'll call the radius of the circle, which is the distance between O and C, b, which, for a semicylinder, equals $4a/(3\pi)$ (see Chapter 1), where a is the radius of the semicylinder. I have marked, in red, the three forces on the semicylinder, and also, in green, the radial and transverse components of the acceleration.

The angular speed $\dot\theta$ can be obtained from energy considerations. The gain in kinetic energy in going from rest to an angular speed $\dot\theta$ is $\frac{1}{2}(mk^2)\dot\theta^2$, and the gain in potential energy when the centre of mass drops through a vertical distance $b \sin\theta$ is $mgb\sin\theta$. Here k is the radius of gyration about O, which, for a semicylinder, is given by $k^2 = \frac{1}{2}a^2$.

[I have left b and k as they are in the equations, so that the analysis could easily be adapted, if needed, for a hollow semicylinder, or a solid hemisphere, or a hollow hemisphere. From Chapters 1 and 2 we recall:

Solid semicylinder:	$b = \dfrac{4a}{3\pi}$	$k^2 = \frac{1}{2}a^2$	$\dfrac{b^2}{k^2} = \dfrac{32}{9\pi^2}$
Hollow semicylinder:	$b = \dfrac{2a}{\pi}$	$k^2 = a^2$	$\dfrac{b^2}{k^2} = \dfrac{4}{\pi^2}$
Solid hemisphere:	$b = \dfrac{3a}{8}$	$k^2 = \frac{2}{5}a^2$	$\dfrac{b^2}{k^2} = \dfrac{45}{128}$
Hollow hemisphere:	$b = \frac{1}{2}a$	$k^2 = \frac{2}{3}a^2$	$\dfrac{b^2}{k^2} = \dfrac{3}{8}$]

On equating the gain in kinetic energy to the loss in potential energy, we obtain

$$\dot\theta^2 = \frac{2bg}{k^2}\sin\theta. \qquad (1)$$

The angular acceleration $\ddot\theta$ can be obtained from applying $\tau = I\ddot\theta$ about O:

$$mgb\cos\theta = mk^2\ddot\theta,$$

from which

$$\ddot\theta = \frac{bg}{k^2}\cos\theta. \qquad (2)$$

The horizontal and vertical equations of motion are

$$N_2 = mb(\dot{\theta}^2 \cos\theta + \ddot{\theta}\sin\theta) \qquad (3)$$

and
$$N_1 - mg = mb(\dot{\theta}^2 \sin\theta - \ddot{\theta}\cos\theta). \qquad (4)$$

We don't really need equation (4), because we are trying to determine when $N_2 = 0$.

On substitution from equations (1) and (2), equation (3) becomes

$$N_1 = \frac{6mb^2 g}{a^2}\sin\theta\cos\theta. \qquad (5)$$

This is zero when $\theta = 0^\text{o}$ (which was the initial condition) or when $\theta = 90^\text{o}$, at which point contact with the wall is lost, which it was required to show.

At this instant, the rotational velocity is $\sqrt{\frac{2bg}{k^2}}$ counterclockwise.

and the linear velocity of C is $b\sqrt{\frac{2bg}{k^2}}$ horizontally to the right.

The rotational kinetic energy is $\frac{1}{2}I\omega^2$, where $\omega = \sqrt{\frac{2bg}{k^2}}$, and I is the rotational inertia *about the centre of mass*, which is $m(k^2 - b^2)$.

$$\therefore \qquad K_\text{rot} = \frac{mbg(k^2 - b^2)}{k^2}.$$

The translational kinetic energy is $\frac{1}{2}mv^2$, where $v = \sqrt{\frac{2bg}{k^2}}$.

$$\therefore \qquad K_\text{tr} = \frac{mb^3 g}{k^2}.$$

The sum of these is mbg, which is just equal to the loss of the original potential energy, which serves as a check on the correctness of our algebra.

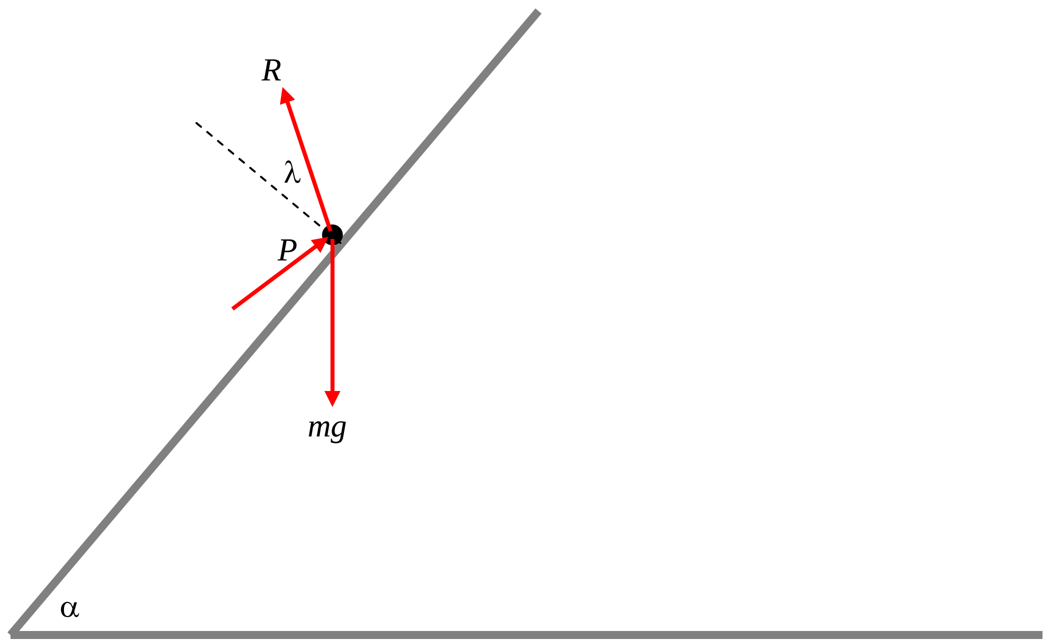
There are now no horizontal forces, so the horizontal component of the velocity of C remains constant. The semicylinder continues to rotate, however, until the rotational kinetic energy is converted to potential energy and C rises to its maximum height. If the base then makes an angle ϕ with the vertical, the gain in potential energy is mbg sin ϕ, and equating this to the rotational kinetic energy gives

$$\sin\phi = 1 - b^2/k^2 .$$

This gives the following results:

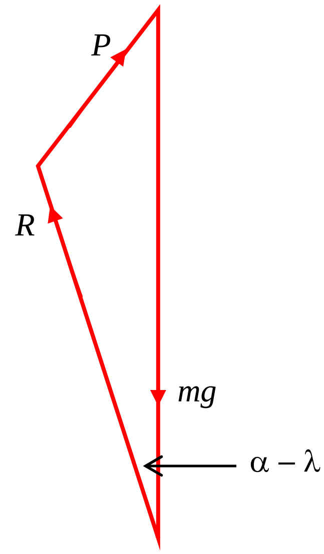
Solid semicylinder:	$\phi = 39^\circ\ 46'$
Hollow semicylinder:	$\phi = 36^\circ\ 30'$
Solid hemisphere:	$\phi = 40^\circ\ 25'$
Hollow hemisphere:	$\phi = 38^\circ\ 41'$

10.

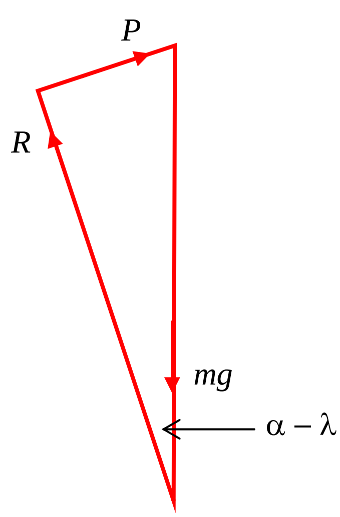


It is well known that if $\alpha > \tan^{-1}\mu$ the particle will slide down the plane unless helped by an extra force. I have drawn the three forces acting on the particle. Its weight *mg*. The reaction *R* of the plane on the particle; if the particle is in limiting static equilibrium, this reaction will make an angle λ ("the angle of friction") with the plane such that $\tan\lambda = \mu$. It therefore makes an angle $\alpha - \theta$ with the vertical. Finally, the additional force *P* needed; we do not initially know the direction of this force.

When three (or more) coplanar forces are in equilibrium and are drawn head-to-tail, they form a closed triangle (polygon). I draw the triangle of forces below.



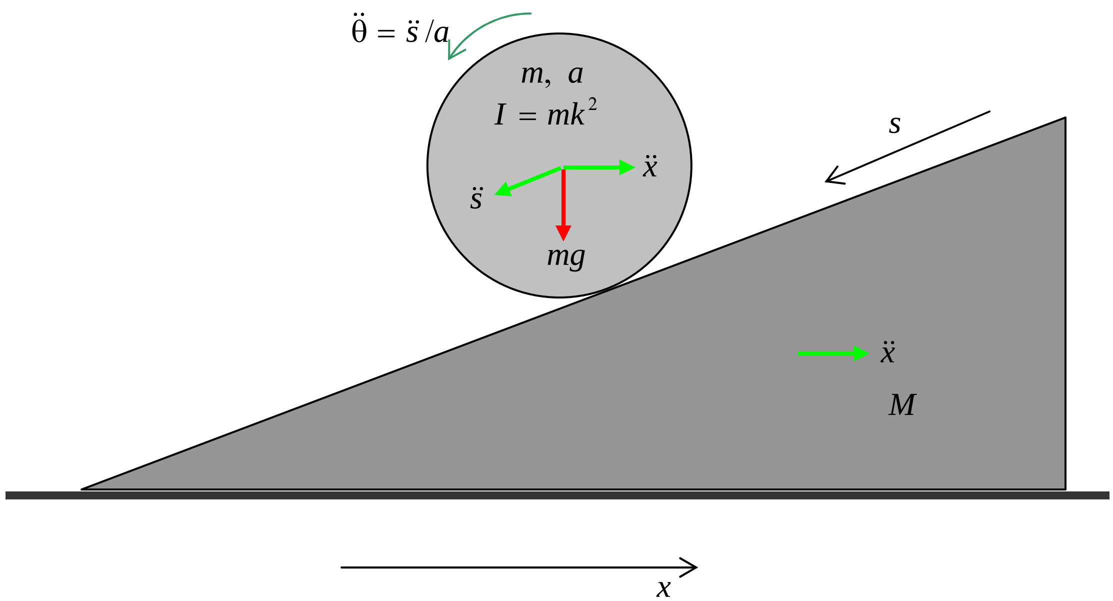
It will be clear from the triangle that P is least when the angle between **P** and **R** is 90°:



The least value of P is therefore $mg(\sin\alpha\cos\lambda - \cos\alpha\sin\lambda)$. But $\tan\lambda = \mu$ and therefore $\sin\lambda = \dfrac{\mu}{\sqrt{1+\mu^2}}$ and $\cos\lambda = \dfrac{1}{\sqrt{1+\mu^2}}$.

$$\therefore \qquad P_{\min} = \frac{mg(\sin\alpha - \mu\cos\alpha)}{\sqrt{1-\mu^2}}.$$

11.



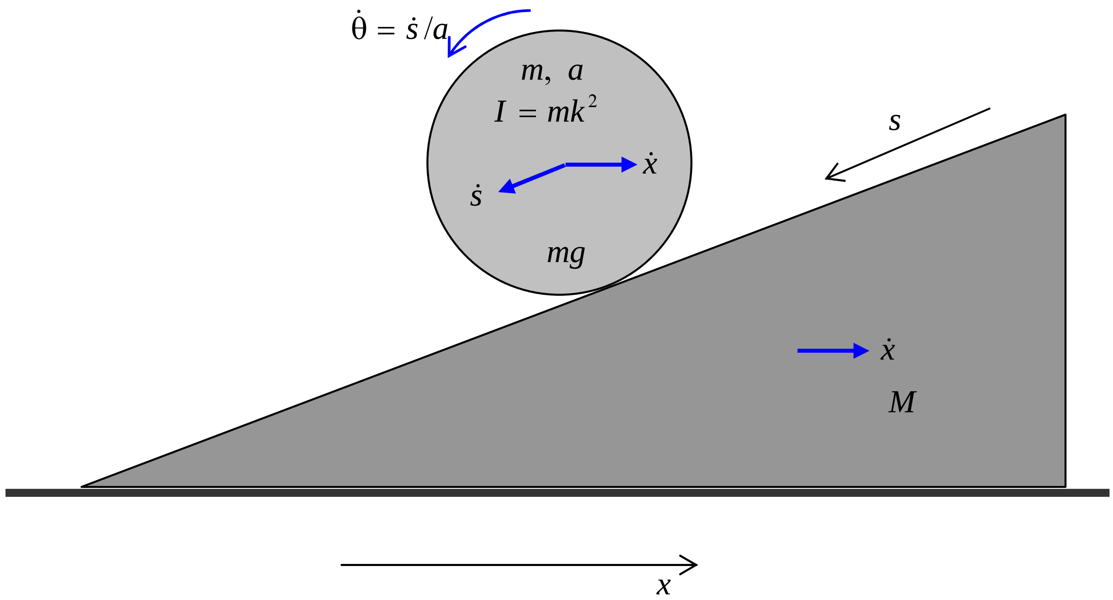
As the cylinder rolls down the plane, the wedge, because its base is smooth, will slide towards the left. Since there are no external horizontal forces on the system, the centre of mass of the system will not move horizontally (or, rather, it won't accelerate horizontally.)

As usual, we draw a large diagram, using a ruler , and we mark in the forces in red and the accelerations in green, after which we'll apply $F = ma$ to the cylinder, or to the wedge, or to the system as a whole, in two directions. It should be easy and straightforward.

I have drawn the linear acceleration $\ddot{s}$ of the cylinder down the slope, and its angular acceleration $\ddot{\theta}$. I have drawn the linear acceleration $\ddot{x}$ of the wedge, which is also shared with the cylinder. I have drawn the gravitational force *mg* on the cylinder. There is one more force on the cylinder, namely the reaction of the wedge on the cylinder. But I'm not sure in which direction to draw it. Is it normal to the plane? That would mean there is no frictional force between the cylinder and the plane. Is that correct (remembering that both the cylinder and the wedge are accelerating? Of course I could calculate the moment of the force *mg* about the point of contact of the cylinder with the plane, and then I wouldn't need to concern myself with any forces at that point of contact. But then that point of contact is not fixed. Oh, dear, I'm getting rather muddled and unsure of myself.

This problem, in fact, is ideally suited to a lagrangian rather than a newtonian treatment, and that is what we shall do. Lagrange proudly asserted that it was not necessary to draw any diagrams in mechanics, because it could all be done analytically. We are not quite so talented a Lagrange, however, so we still need a large diagram drawn with a ruler.

But, instead of marking in the forces and accelerations in red and green, we mark in the *velocities* in *blue*.



No frictional or other nonconservative forces do any work, so we can use Lagrange's equations of motion for a conservative holonomic system; $\frac{d}{dt}\left(\frac{\partial T}{\partial \dot q}\right) - \left(\frac{\partial T}{\partial q}\right) = -\left(\frac{\partial V}{\partial q}\right)$.

The speed of the wedge is $\dot x$ and the speed of the centre of mass of the cylinder is $\sqrt{\dot s^2 + \dot x^2 - 2\dot s\dot x\cos\alpha}$, and the angular speed of the cylinder is $\dot s/a$.

The kinetic energy of the system is

$$T = \tfrac{1}{2}m(\dot s^2 + \dot x^2 - 2\dot s\dot x\cos\alpha) + \tfrac{1}{2}(mka^2)\left(\frac{\dot s}{a}\right)^2 + \tfrac{1}{2}M\dot x^2,$$

or

$$T = \tfrac{1}{2}m\left(1 + \frac{k^2}{a^2}\right)\dot s^2 - m\dot s\dot x\cos\alpha + \tfrac{1}{2}(m + M)\dot x^2,$$

and the potential energy is

$$V = \text{constant} - mgs\sin\alpha.$$

Application of Lagrange's equation to the coordinate *x* gives us

$$m\ddot{s}\cos\alpha = (m + M)\ddot{x}$$

and application of Lagrange's equation to the coordinate *s* gives us

$$\left(1 + \frac{k^2}{a^2}\right)\ddot{s} = \ddot{x}\cos\alpha + g\sin\alpha .$$

Elimination of $\ddot{s}$ from these two equations gives us

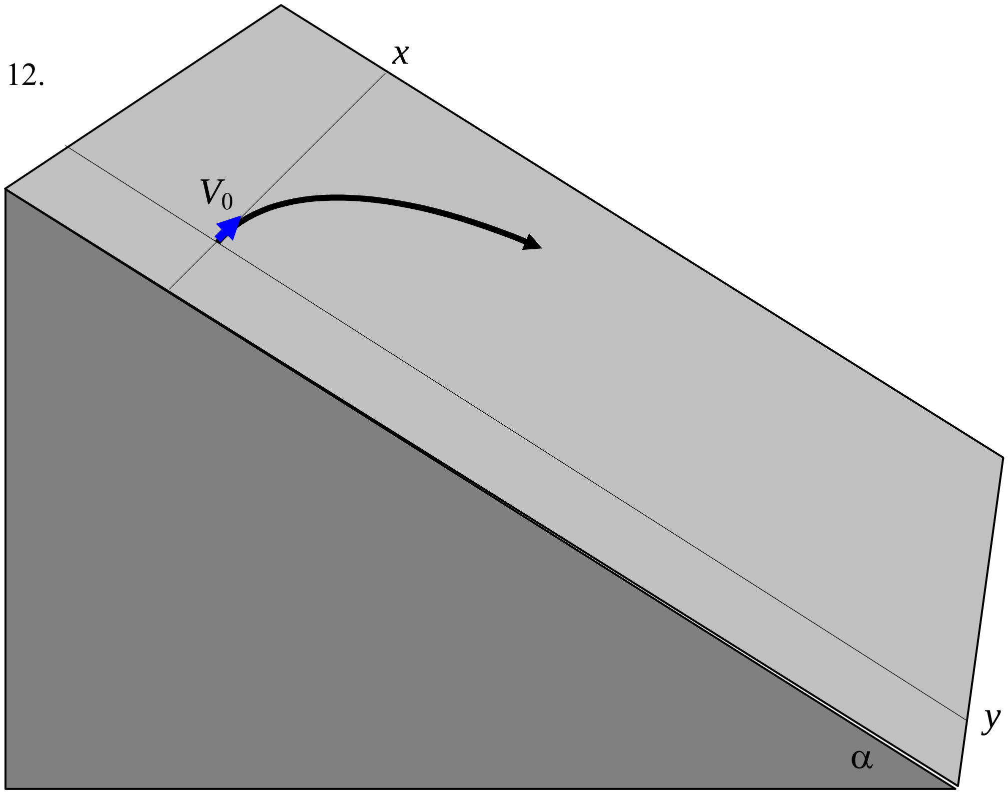
$$\ddot{x} = \frac{mg\sin\alpha\cos\alpha}{(m+M)\left(1 + \frac{k^2}{a^2}\right) - m\cos^2\alpha} .$$

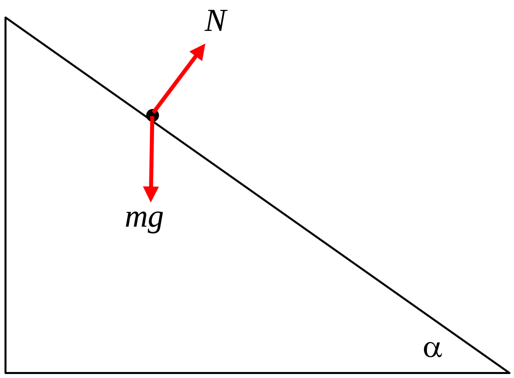
You can also easily find an expression for $\ddot{s}$ is you wish.

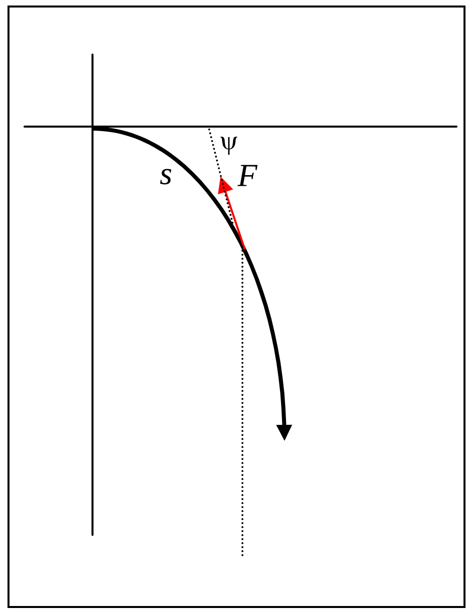
If the wedge had been fixed and immovable, and not free to slide on a smooth table, you would easily have been able to obtain $\ddot{s}$ by newtonian methods. If you put $M = \infty$ in the expression for $\ddot{s}$ for the smooth table, do you get the same as you do for the immovable wedge? (You *should* do.)

Now that you have found $\ddot{x}$, you can also find $\ddot{s}$ and $\ddot{\theta}$. You therefore now know the magnitude and direction of the velocity of the centre of mass of the cylinder. From this you should be able to find the net force and torque on the cylinder, and from these, you should be able to find the magnitude and direction of the reaction of the wedge on the cylinder. I have not pursued this. If anyone succeeds in doing so, and would like his/her solution poster here, with acknowledgment, let me know.

12.







There is no acceleration normal to the plane, and therefore $N = mg\cos\alpha$. The frictional force F acts along the tangent to the path and is equal to μN, or $\mu mg \cos \alpha$, where μ is the coefficient of moving friction. We are told to ignore the difference between the coefficients of moving and limiting static friction. Since the particle was originally at rest in limiting static friction, we must have $\mu = \tan\alpha$. Therefore $F = mg\sin\alpha$. The tangential equation of motion is

$m\ddot{s} = -F +$ whatever the component of $m\mathbf{g}$ is in the tangential direction in the sloping plane.

The component of $m\mathbf{g}$ down the plane would be (look at the left hand drawing) $mg\sin\alpha$, and so its tangential component (look at the right hand drawing) is $mg\sin\alpha\sin\psi$. So we have, for the tangential equation of motion,

$$m\ddot{s} = -mg\sin\alpha + mg\sin\alpha\sin\psi,$$

or

$$\ddot{s} = -g\sin\alpha(1-\sin\psi).$$

We are seeking a relation between V and ψ, so, in the now familiar fashion, we write $V\dfrac{dV}{ds}$ for $\ddot{s}$, so the tangential equation of motion is

$$V\frac{dV}{ds} = -g\sin\alpha(1-\sin\psi). \qquad (1)$$

We also need the equation of motion normal to the trajectory. The component of $m\mathbf{g}$ in that direction is $mg\sin\alpha\cos\psi$, and so the normal equation of motion is

$$\frac{mV^2}{\rho} = mg\sin\alpha\cos\psi.$$

Here ρ is the radius of curvature of the path, which is the reciprocal of the curvature $ds/d\psi$. The normal equation of motion is therefore

$$V^2\frac{d\psi}{ds} = g\sin\alpha\cos\psi. \qquad (2)$$

Divide equation (1) by equation (2) to eliminate s and thus get a desired differential equation between V and ψ:

$$\frac{1}{V}\frac{dV}{d\psi} = -\frac{(1-\sin\psi)}{\cos\psi}. \qquad (3)$$

This is easily integrated; a convenient (not the only) way is to multiply top and bottom by $1 + \sin\psi$. In any case we soon arrive at

$$\ln V = -\ln(1 + \sin\psi) + \text{constant}, \qquad (4)$$

and with the initial condition $V = V_0$ when $\psi = 0$, this becomes

$$V = \frac{V_0}{1+\sin\psi}. \qquad (5)$$

In the limit, as $\psi \to 90^\circ$, $V \to \frac{1}{2}V_0$. The particle is then moving at constant velocity and is in equilibrium under the forces acting upon it just when it was initially at rest.

13. M_1 = mass of complete sphere of radius a.
M_2 = mass of missing inner sphere of radius xa.
M = mass of given hollow sphere.

We have $M = M_2 - M_1$ and $M_2/M_1 = x^3$, and therefore

$$M_1 = \frac{M}{1-x^3} \text{ and } M_2 = \frac{Mx^3}{1-x^3}.$$

Also $I = \frac{2}{5}M_1a^2 - \frac{2}{5}M_2x^2a^2 = \frac{2}{5}a^2(M_1 - M_2x^2)$.

Hence
$$I = \frac{2}{5}Ma^2 \times \frac{1-x^5}{1-x^3}.$$

If $x = 0$, $I = \frac{2}{5}Ma^2$, as expected. If $x \to 1$, you may have to use de l'Hôpital's rule to show that $I \to \frac{2}{3}Ma^2$, as expected.

14. M_1 = mass of mantle.
M_2 = mass of core
M = mass of entire planet.

We have $M = M_1 + M_2$ and $\frac{M_1}{M_2} = \frac{s(1-x^3)}{x^3}$, and therefore

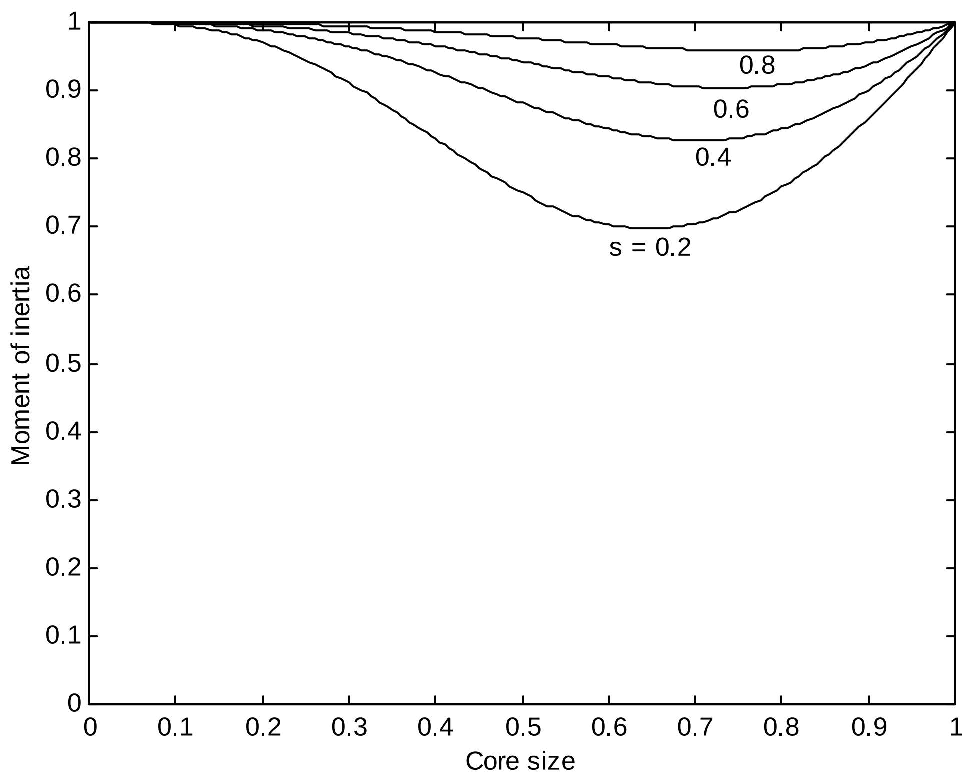
$$M_2 = M \times \frac{x^3}{x^3 + s(1-x^3)} \quad \text{and} \quad M_1 = M \times \frac{s(1-x^3)}{x^3 + s(1-x^3)}.$$

Also $$I = I_{\text{core}} + I_{\text{mantle}} = \tfrac{2}{5} M_2 x^2 a^2 + \tfrac{2}{5} M_1 a^2 \times \frac{1 - x^5}{1 - x^3},$$

where I have made use of the result from the previous problem. On substitution of the expressions for M_1 and M_2, we quickly obtain

$$I = \tfrac{2}{5} Ma^2 \times \frac{s + (1 - s)x^5}{s + (1 - s)x^3}. \qquad (1)$$

A hollow planet would correspond to $1/s = 0$. Divide top and bottom by s and it is immediately seen that the expression for a hollow planet would be identical to the expression obtained for the previous problem.



Note that both $x = 0$ and $x = 1$ correspond to a uniform sphere, so that in either case, $I = \tfrac{2}{5} Ma^2$; for all other cases, the moment of inertia is less than $\tfrac{2}{5} Ma^2$.

The core size for minimum moment of inertia is easily found by differentiation of the above expression for I, and the required expression follows after some algebra. For s = 0.6, the equation becomes $9 - 15x^2 - 4x^5 = 0$, of which the only positive real root is $x = 0.736382$, which corresponds to a moment of inertia of $0.90376 \times \tfrac{2}{5} Ma^2$. Note that. for $s = 0.6$, the moment of inertia, expressed in units of $\tfrac{2}{5} Ma^2$, varies very little as the core size goes from 0 to 1, so that measurement of the moment of inertia places very little restriction on the possible core size.

The inverse of equation (1) is

$$(1-s)x^5 - I(1-s)x^3 + (1-I)s = 0, \qquad (2)$$

where I is expressed in units of $\frac{2}{5}Ma^2$. For $I = 0.911$, there are two positive real roots (look at the graph); they are $x = 0.64753$ and 0.81523. For $I = 0.929$, the roots are 0.55589 and 0.87863. Thus the core size could be anything between 0.55589 and 0.64753 or between 0.81523 and 0.87863 a rather large range of uncertainty. Even if I were known exactly (which does not happen in science), there would be two solutions for x.

15. This is just a matter of geometry. If, when you make a small angular displacement, you raise the centre of mass of the brick the equilibrium is stable. For, while the brick is in its vertical position, it is evidently at a potential minimum, and you have to do work to raise the centre of mass. If, on the other hand, your action in making a small angular displacement results in a lowering of the centre of mass, the equilibrium is unstable.

When the brick is in its vertical position, the height h_0 of its centre of mass above the base of the semicylinder is just

$$h_0 = R + l.$$

When it is displaced from the vertical by an angle θ, the point of contact between brick and semicylinder is displaced by a distance $R\theta$, and, by inspection of the drawing, the new height h is

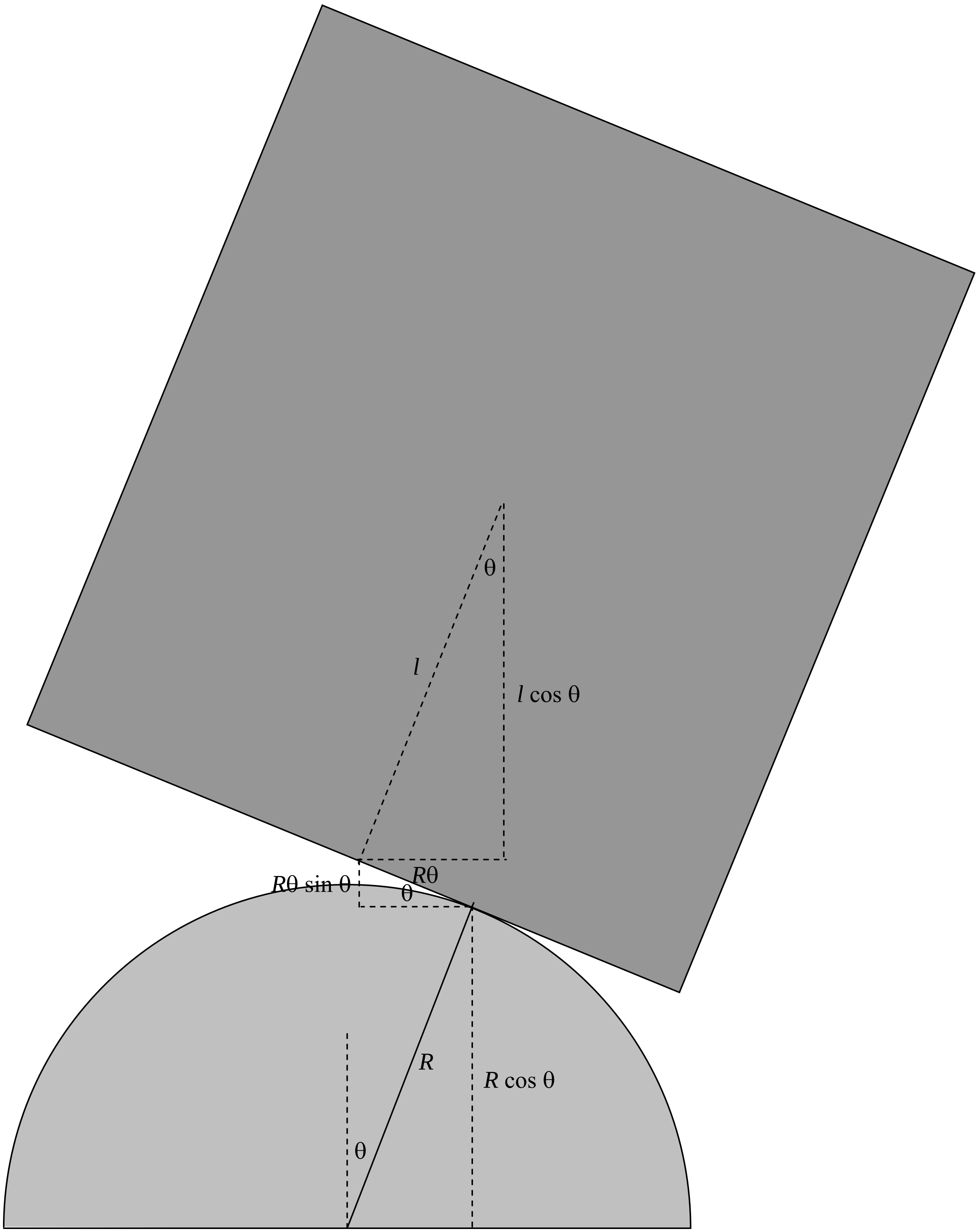
$$h = R\cos\theta + R\theta\sin\theta + l\cos\theta.$$

$$\therefore \qquad h - h_0 = R\theta\sin\theta - (R+l)(1-\cos\theta).$$

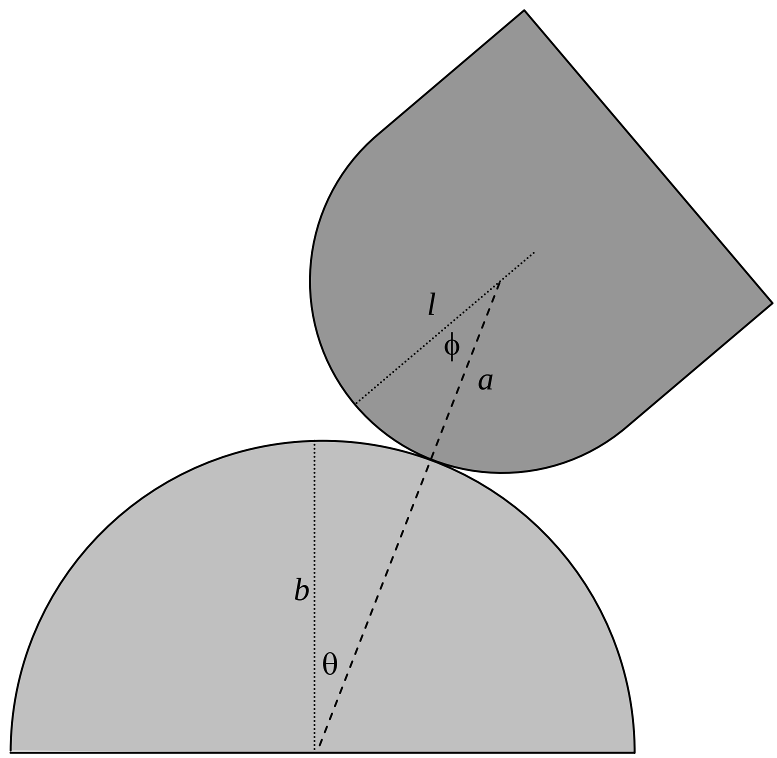
If you Maclaurin expand this as far as θ^2, you arrive at

$$h - h_0 \approx \tfrac{1}{2}(R-l)\theta^2.$$

This is positive, and therefore the equilibrium is stable, if $l < R$, or $2l < 2R$, i.e. if the length of the brick is less than the diameter of the semicylinder.


θ
l
l cos θ
Rθ sin θ
Rθ
θ
R
R cos θ
θ

16. As in the previous question, it is just a matter of geometry. If rolling the Thing results in raising its centre of mass, the equilibrium is stable. Initially, the height of the centre of mass is $h_0 = b + l$.



After rolling, the dashed line, which joins the centres and is of length $a + b$, makes an angle θ with the vertical. The short line joining the centre of mass of the Thing to the centre of curvature of its bottom is of length $l - a$ and it makes an angle $\theta + \phi$ with the vertical. The height of the centre of mass is therefore now

$$h = (a+b)\cos\theta + (l-a)\cos(\theta+\phi).$$

The centre of mass has therefore rise through a height

$$h - h_0 = (a+b)\cos\theta + (l-a)\cos(\theta+\phi) - b - l .$$

Also, the two angles are related by $a\phi = b\theta$, so that

$$h - h_0 = (a+b)\cos\theta + (l-a)\cos[\{1 + (b/a)\}\theta] - b - l\ .$$

Maclaurin expand the cosines to θ^2 and you should get

$$h - h_0 = -\tfrac{1}{2}\theta^2[a + b + (l-a)(1 + b/a)^2]\ .$$

For stability this must be positive, and hence $\dfrac{1}{l} > \dfrac{1}{a} + \dfrac{1}{b}$.

If $a = b$, this becomes $l < \tfrac{1}{2}a$.

For a hollow semicylinder, $l = (1 - 2/\pi)a = 0.363a$. ∴ Stable

For a hollow hemisphere, $l = 0.5a$. ∴ Borderline stable

For a solid semicylinder, $l = [1 - 4/(3\pi)]a = 0.576a$. ∴ Unstable

For a solid hemisphere, $l = \tfrac{5}{8}a = 0.625a$. ∴ Unstable

17. We need to find the height h of the centre of mass above the level of the pegs as a function of θ . See drawing on next page.

Angles:

BAC $= 45^\circ - \theta$
ABX $= 45^\circ + \theta$

Distances:

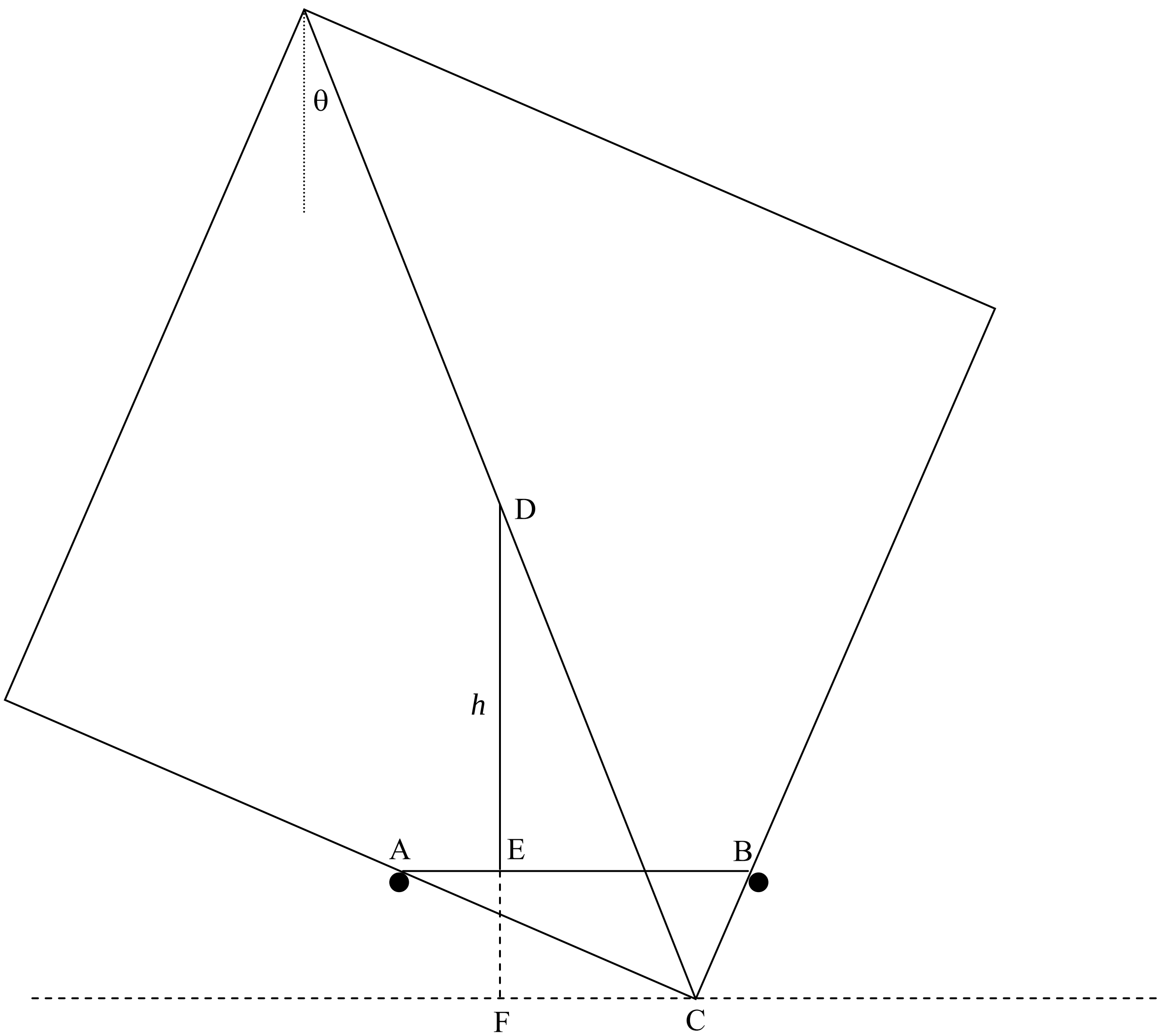
AB $= 2ac$
AC $= 2ac \cos(45^\circ - \theta)$
EF $= 2ac \cos(45^\circ - \theta)\cos(45^\circ + \theta) = ac \cos 2\theta$
DC $= a\sqrt{2}$
DF $= a\sqrt{2}\cos\theta$
h = DF − EF $= a(\sqrt{2}\cos\theta - c\cos 2\theta)$
h_0 = height of centre of mass above pegs when $\theta = 0^\circ = a(\sqrt{2} - c)$

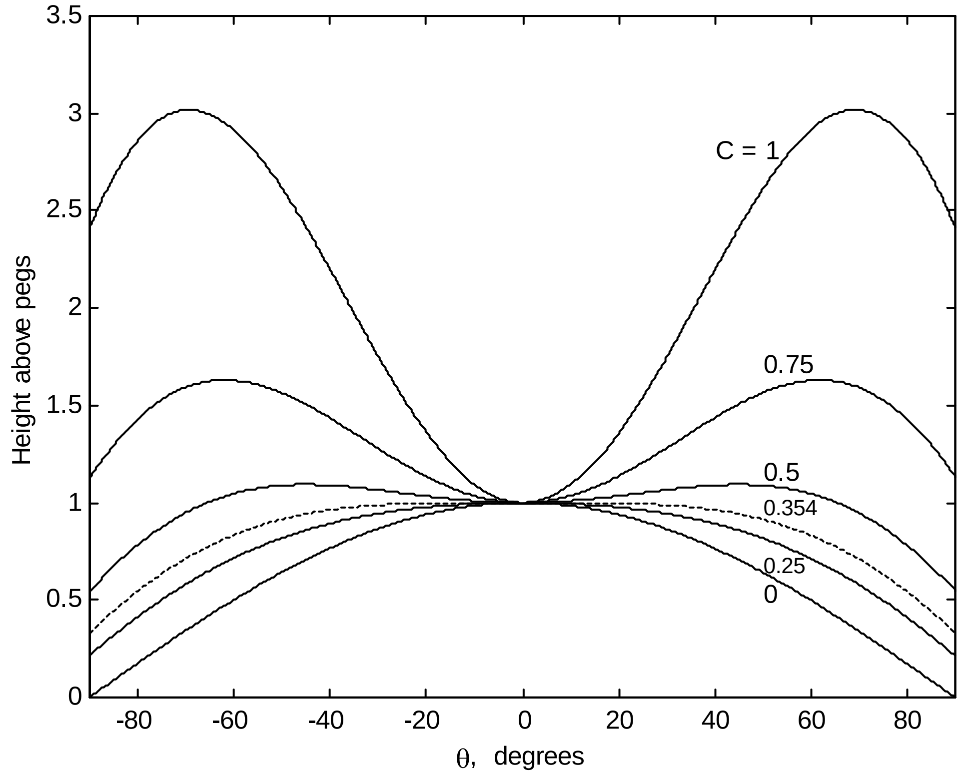
$$y = \frac{h}{h_0} = \frac{\sqrt{2}\cos\theta - c\cos 2\theta}{\sqrt{2} - c}$$

$dy/d\theta$ will show that maxima and minima of y (and of the potential energy), and hence equilibria, occur for $\theta = 0^\circ$ and for $\cos\theta = 1/(c\sqrt{8})$, which is possible only if $c > 1/\sqrt{8}$.

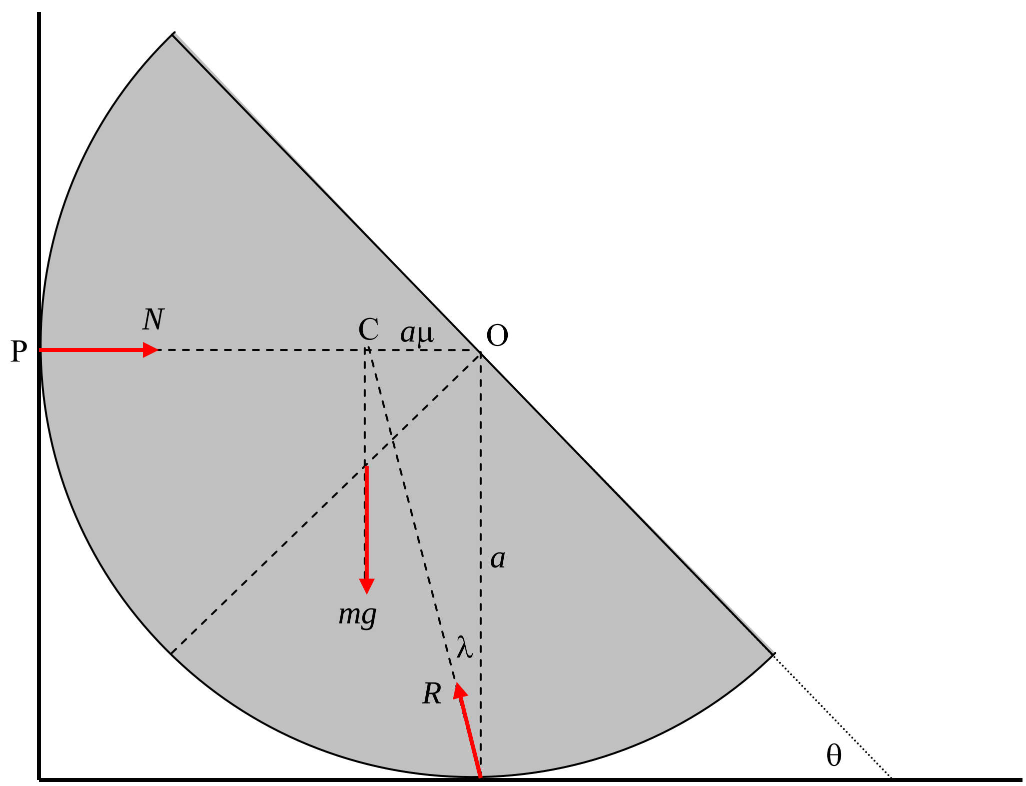
A second differentiation will show which extrema are maxima and which are minima. In particular the second derivative at $\theta = 0^{\circ}$ is zero for $c = 1/\sqrt{8}$. I draw graphs of $y : \theta$ for several *c* below.

If $c = \frac{1}{2}$, $\theta = 45^{\circ}$.




3.5
3
2.5
2
1.5
1
0.5
0
Height above pegs
-80
-60
-40
-20
0
20
40
60
80
θ, degrees
C = 1
0.75
0.5
0.354
0.25
0

18.

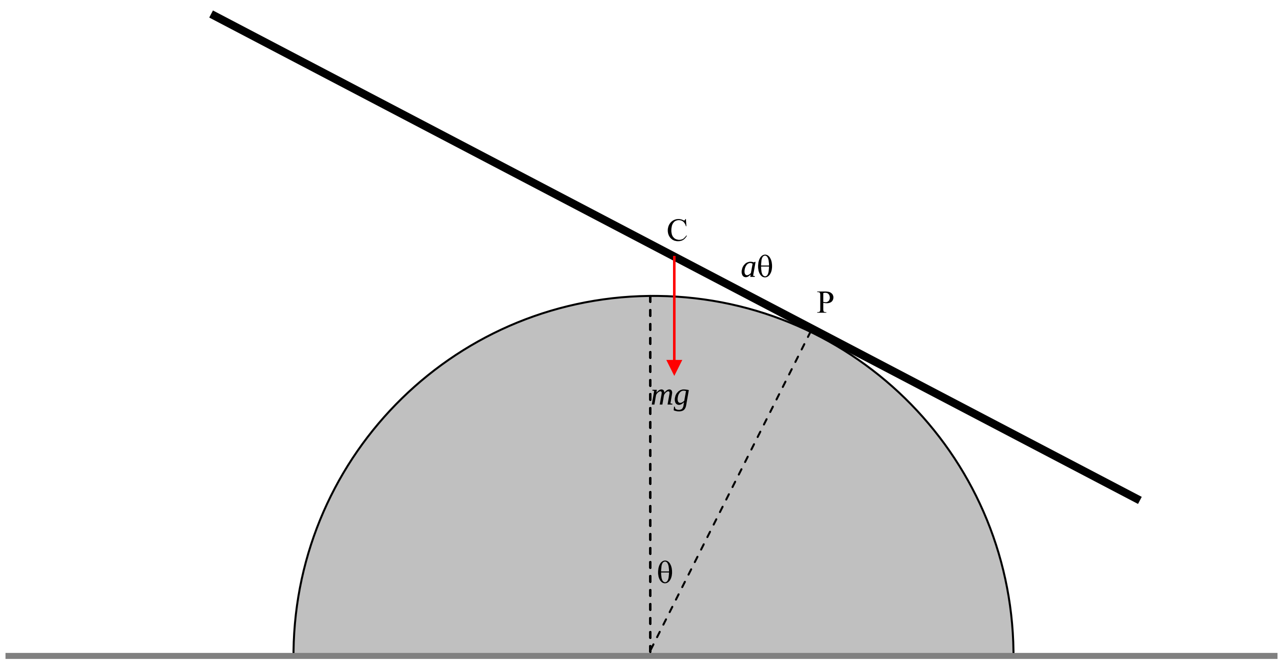

N
P
C
aμ
O
mg
a
λ
R
θ

There are three forces acting on the hemisphere: Its weight *mg*. The reaction *N* of the wall, which is perpendicular to the wall since the wall is smooth. The reaction *R* of the floor, which acts at an angle λ to the floor, where $\mu = \tan\lambda$. Three forces in equilibrium must act through a point; therefore all three forces act through the point P. It is thus clear that

$$\sin\theta = \frac{\mathrm{OP}}{\mathrm{OC}} = \frac{a\mu}{\frac{3}{8}a} = \frac{8\mu}{3}.$$

If $\mu = \frac{1}{4}$, $\theta = 41^\circ 48'$. If $\mu = \frac{3}{8}$, $\theta = 90^\circ$. If $\mu > \frac{3}{8}$, the hemisphere can rest in any position, the equilibrium not being *limiting* static equilibrium.

19.



When the rod makes an angle θ to the horizontal, the distance from its centre of mass C to the point of contact P is $a\theta$. The moment of inertia about P is $\frac{1}{3}ml^2 + m(al)^2$. The torque about P is $mga\theta\cos\theta$. The equation of motion is

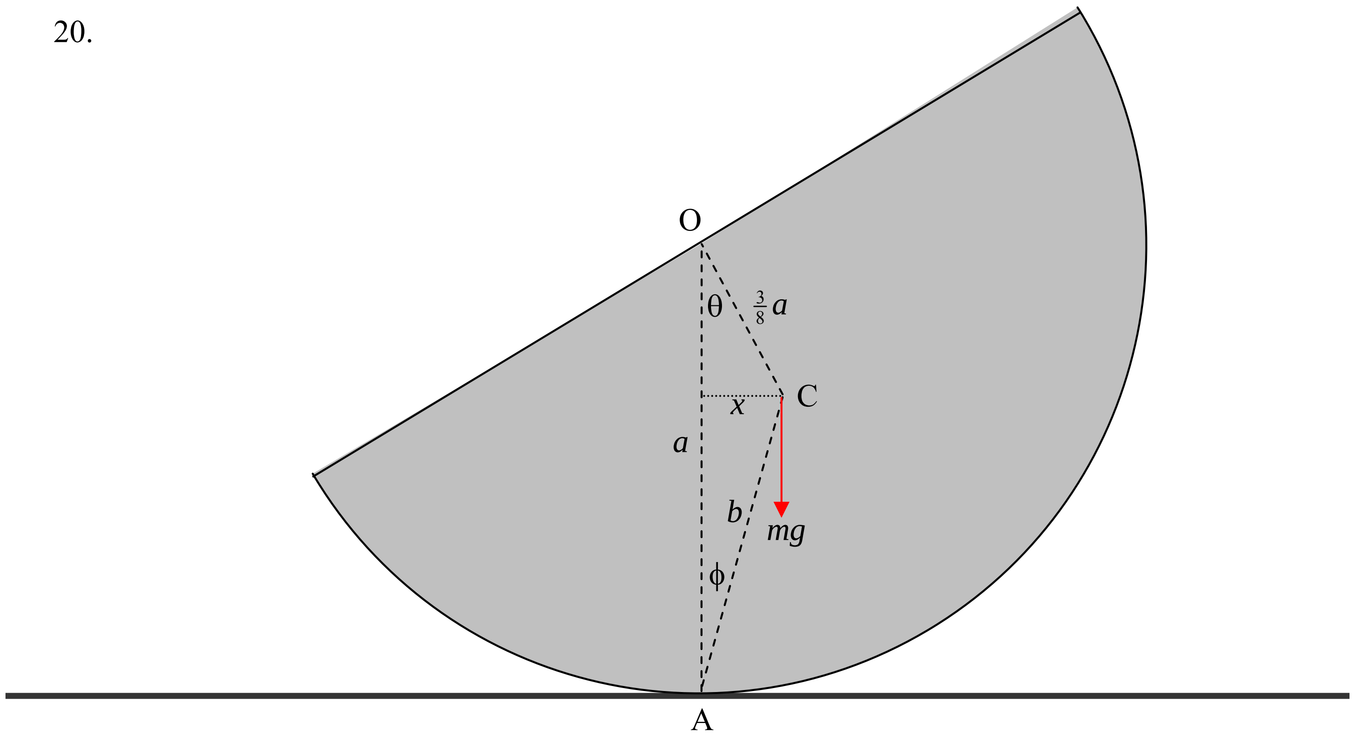
$$m(\tfrac{1}{3}l^2 + a^2\theta^2)\ddot{\theta} = -mga\theta\cos\theta.$$

To first order in θ, $\cos\theta \approx 1$ and $a^2\theta^2 << \frac{1}{3}l^2$. Therefore, to first order, the equation of motion is

$$\tfrac{1}{3}l^2\ddot{\theta} = -ga\theta$$

and so the period is $$P = \frac{2\pi l}{\sqrt{3ga}}.$$

20.



I have drawn only one force. I am going to consider the rotational equation of motion about A, so I shan't be concerned with the force at A. The equation of motion about A is

$$I\ddot{\phi} = -mgx,$$

where I is the moment of inertia about A. The moments of inertia about O, C and A, respectively:

$$\tfrac{2}{5}ma^2, \qquad \tfrac{2}{5}ma^2 - m\left(\tfrac{3}{8}a\right)^2, \qquad \tfrac{2}{5}ma^2 - m\left(\tfrac{3}{8}a\right)^2 + mb^2,$$

where I have made good use of the parallel axes theorem. The distance b is given by the cosine rule to triangle OCA, and hence we find, after a little algebra, for the moment of inertia about A:

$$I = ma^2\left(\tfrac{7}{5} - \tfrac{3}{4}\cos\theta\right).$$

We also need

$$x = \frac{3a\sin\theta}{8},$$

and hence we obtain for the equation of motion

$$a(56 - 30\cos\theta)\ddot{\phi} = -15g\sin\theta.$$

For small angles, this becomes

$$26a\ddot{\phi} = -15g\theta.$$

From the sine rule we have

$$\frac{\sin\phi}{\frac{3}{8}a} = \frac{\sin(\theta+\phi)}{a},$$

and replacing the sines by the angles, for small angles, we find that $\phi \approx \frac{3}{5}\theta$. Thus the equation of motion becomes

$$26a\ddot{\theta} = -25g\theta$$

and so the period is $$P = 2\pi\sqrt{\frac{26a}{25g}}.$$

21. The (second) moment of inertia with respect to the centre (see chapter 2.19 of chapter 2) is

$$\mathrm{I}_{\text{centre}} = 4\pi\rho_0\int_0^a (r^4 - r^5/a)dr = \tfrac{2}{15}\pi\rho_0 a^5.$$

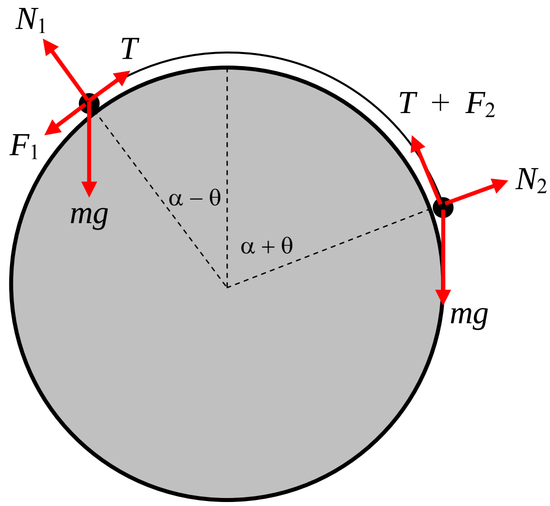
The moment of inertia with respect to an axis through the centre is 2/3 of this:

$$I_{\text{axis}} = \tfrac{4}{45}\pi\rho_0 a^5.$$

$$\therefore \qquad I_{\text{axis}} = \tfrac{4}{15}Ma^2.$$

22.

$N_1 = mg\cos(\alpha-\theta)$
$F_1 = \mu N_1$



$N_2 = mg\cos(\alpha+\theta)$
$F_2 = \mu N_2$

Left-hand particle: $T = mg[\mu\cos(\alpha-\theta) + \sin(\alpha-\theta)].$

Right-hand particle: $T = mg[\sin(\alpha+\theta) - \mu\cos(\alpha+\theta)].$

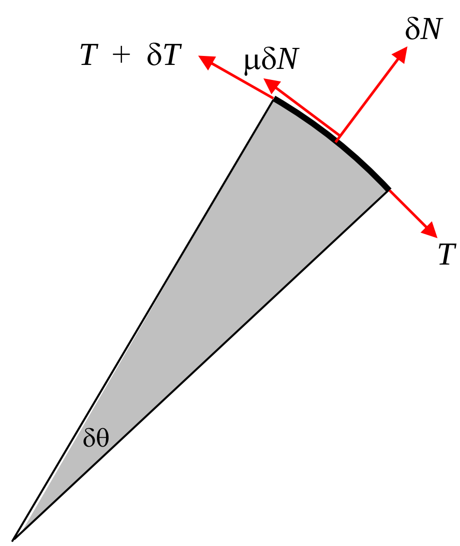
$\therefore \quad \mu[\cos(\alpha-\theta) + \cos(\alpha+\theta)] = \sin(\alpha+\theta) - \sin(\alpha-\theta),$

and, by the "sum and difference" trigonometrical formulae, we obtain

$$2\mu\cos\alpha\cos\theta = 2\cos\alpha\sin\theta,$$

from which $\underline{\underline{\tan\theta = \mu.}}$

23.



Consider a portion of the rope between θ and δθ. There are four forces on this portion. The tension T at θ. The tension $T + \delta T$ at $\theta + \delta\theta$ (δT is negative). The normal reaction δN of the cylinder on the rope. The frictional force $\mu\delta N$ of the cylinder on the rope. Note that the rope is about to slip downwards, so the friction force is upwards as shown.

We have $$\delta N = (2T + \delta T)\sin(\tfrac{1}{2}\theta)$$

and $$(T + \delta T)\cos(\tfrac{1}{2}\delta\theta) + \mu\,\delta N = T\cos(\tfrac{1}{2}\delta\theta).$$

To first order, these become

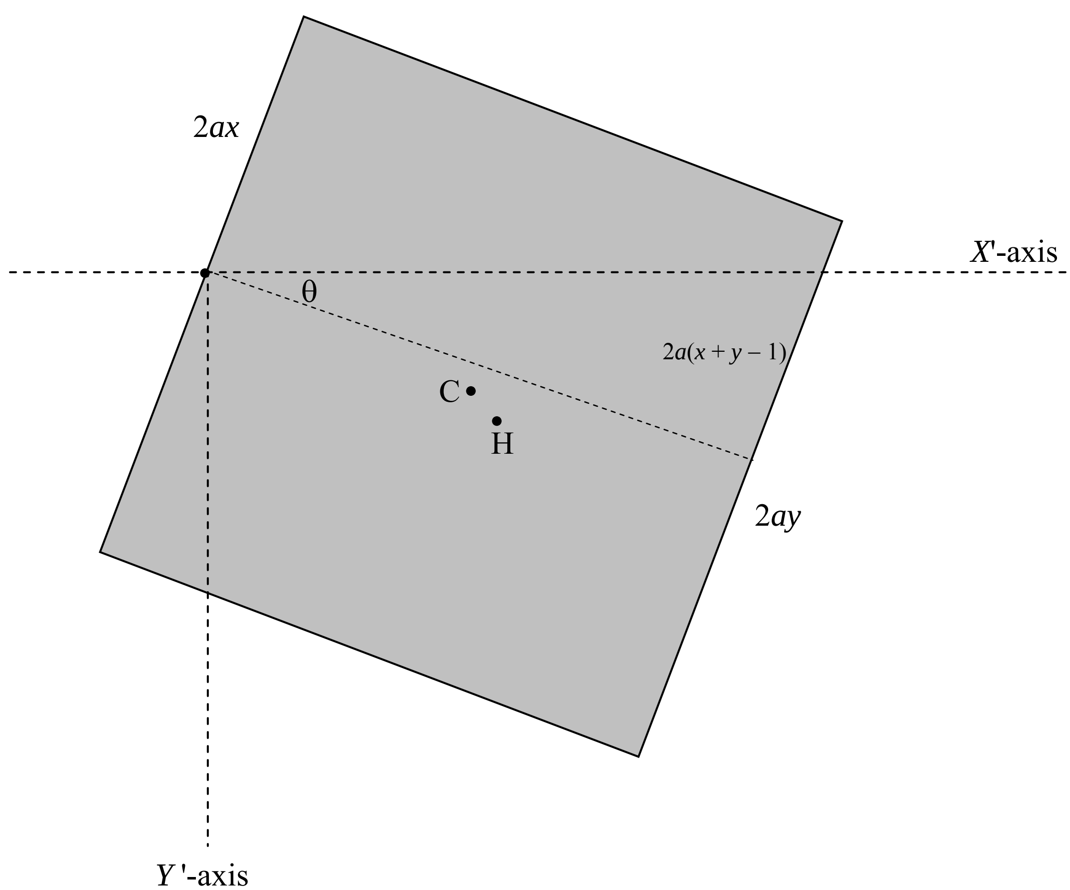
$$\delta N = T\,\delta\theta$$

and $$\delta T = -\mu\,\delta N.$$

∴ $$\delta T = -\mu T\,\delta\theta$$

and hence by integration $$\underline{\underline{F = Mg\,e^{-\mu\alpha}}}.$$

24



. Area of square $= 4a^2$
Area of rectangle $= 4a^2(1-x)$
Area of triangle $= 2a^2(x+y-1)$
Area of trapezoid $= 2a^2(1-x+y)$

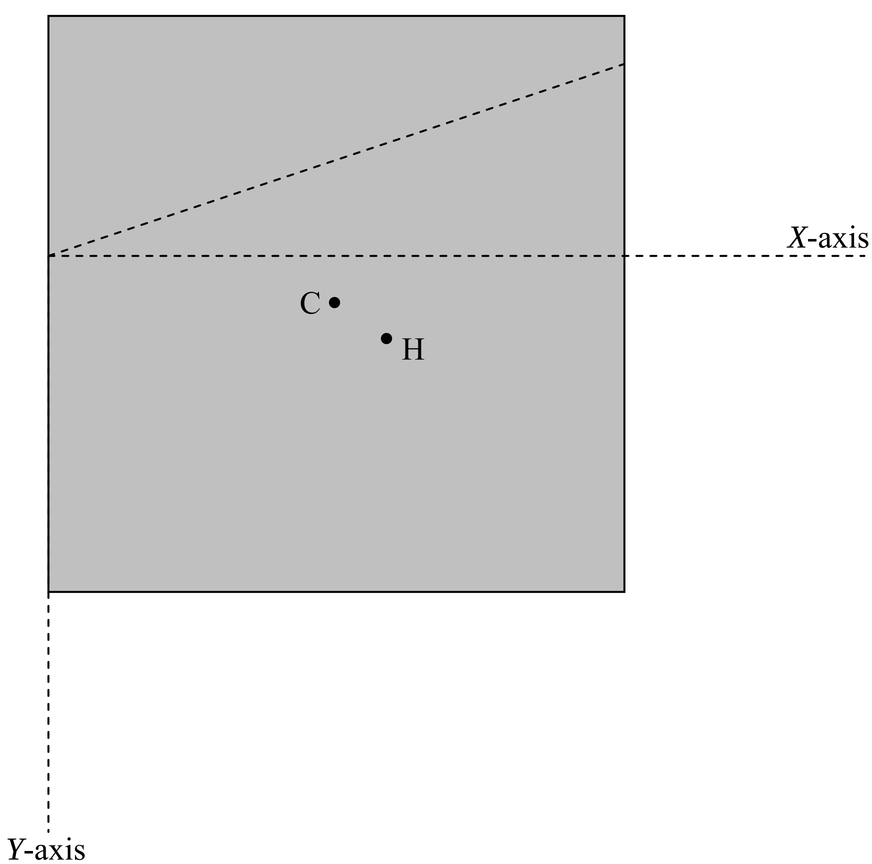
The weight of the cube is $8a^3\rho sg$, and it acts downward through C, the centre of mass. The hydrostatic upthrust is $4a^3(1-x+y)\rho g$ and it acts upward through the centre of buoyancy H. Here ρ is the density of the fluid, and ρs is the density of the wood. We evidently must find the X'- coordinate of C and of H. Let's first of all find the X- and Y- coordinates (see the next figure).

The X- and Y- coordinates of C are trivial and quite easy respectively:

$$X_C = a \qquad Y_C = a(1-2x)$$

You are going to have to work quite hard at it to find the X- and Y- coordinates of H, the centre of buoyancy, which is the centroid of the trapezoid. "After some algebra" you should find

$$X_H = \frac{2(1-x+2y)a}{3(1-x+y)} \qquad Y_H = \frac{2(2-4x+2y+2x^2-2xy-y^2)a}{3(1-x+y)}$$



To find the X'- coordinates of C and of H, we use the usual formulas for rotation of axes, being sure to get it the right way round:

$$\begin{pmatrix} X' \\ Y' \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} X \\ Y \end{pmatrix},$$

together with $\tan\theta = x + y - 1.$

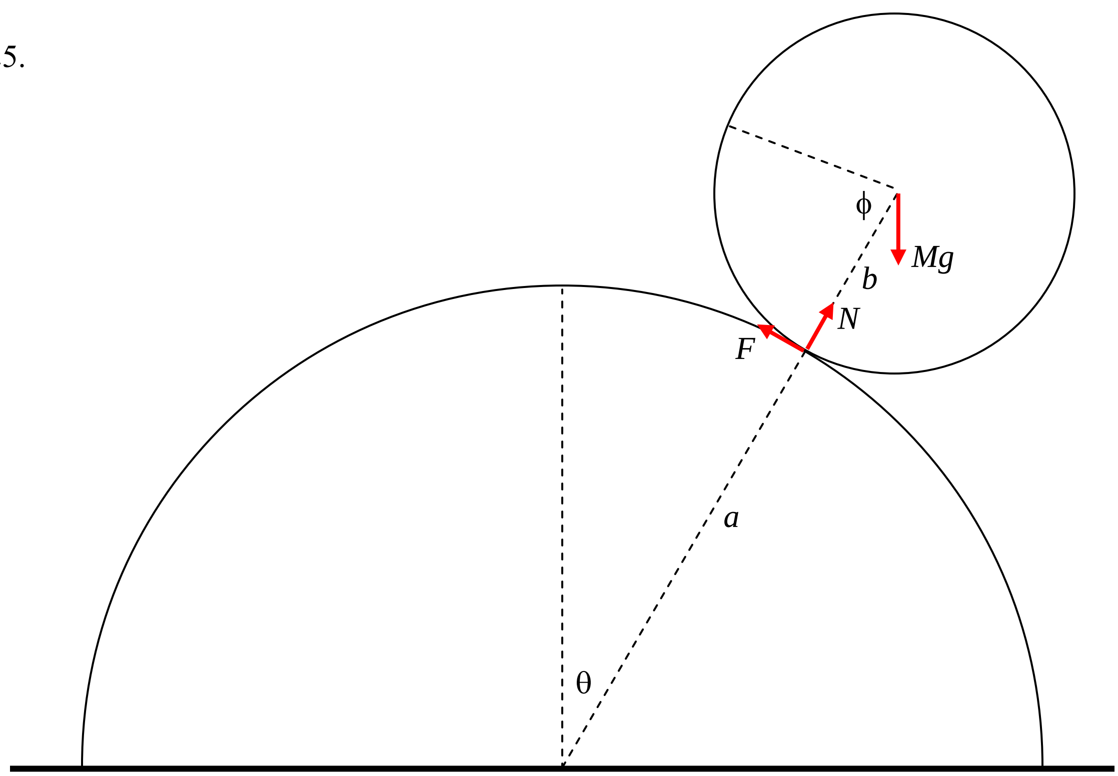
Take moments about the axle (origin):

$$8a^3\rho s g X'_C = 4a^3\rho g(1 - x + y).$$

After a little more algebra, you should eventually arrive at

$$s = \frac{3 - 7x + 2y + 6x^2 - 3y^2 - 2x^3 + y^3 + 3xy^2}{3(2 - 3x - y + 2x^2 + 2xy)}.$$

25.



Let the radii of the cylinder and sphere be a and b respectively, and the mass of the sphere be M. The angles θ and ϕ are related by $a\theta = b\phi$. I have drawn the three forces on the sphere, namely its weight, the normal reaction of the cylinder on the sphere, and the frictional force on the sphere. The transverse acceleration of the centre of the sphere is $(a+b)\ddot{\theta}$ and the centripetal acceleration is $(a+b)\dot{\theta}^2$. The equations of motion are:

$$Mg\sin\theta - F = M(a+b)\ddot{\theta} \tag{1}$$

and

$$Mg\cos\theta - N = M(a+b)\dot{\theta}^2. \tag{2}$$

The angular acceleration of the sphere about its centre is $\ddot{\theta} + \ddot{\phi} = (1 + a/b)\ddot{\theta}$, and its rotational inertia is $2Mb^2/5$. The torque that is causing this angular acceleration is Fb, and therefore the rotational equation of motion is

$$Fb = \tfrac{2}{5}Mb^2(1 + a/b)\ddot{\theta}. \tag{3}$$

Elimination of F between equations (1) and (3) yields

$$\ddot{\theta} = \frac{5g}{7(a+b)}\sin\theta. \tag{4}$$

Write $\ddot{\theta}$ as $\dot{\theta}\,d\dot{\theta}/d\theta$ in the usual way and integrate with initial conditions $\theta = \dot{\theta} = 0$, or from energy considerations:

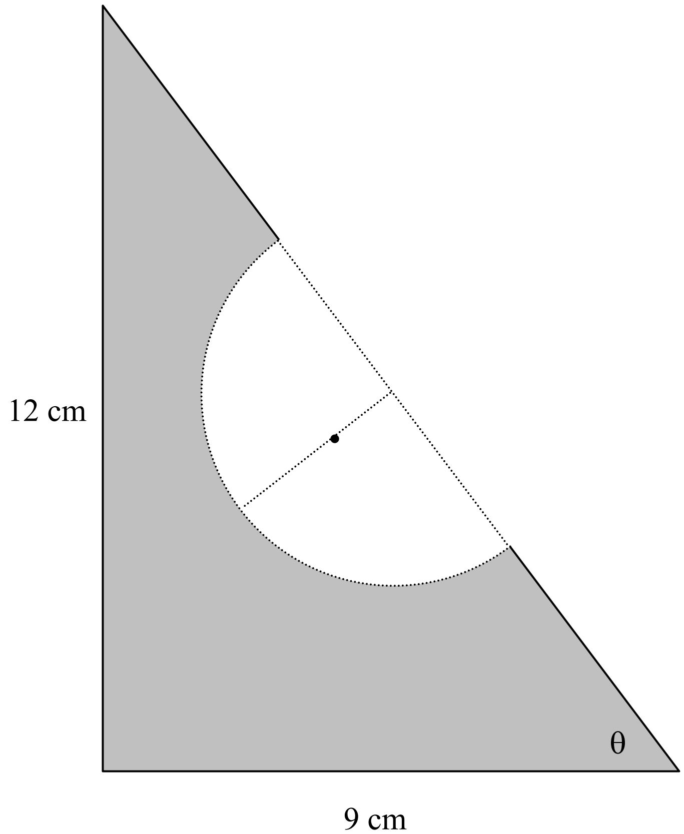
$$\dot{\theta}^2 = \frac{10g}{7(a+b)}(1 - \cos\theta). \tag{5}$$

Substitute for $\ddot{\theta}$ and $\dot{\theta}^2$ into equation (2) to obtain

$$N = Mg(17\cos\theta - 10). \tag{6}$$

This is zero, and the sphere leaves the cylinder, when $\cos\theta = 10/17$, $\theta = 53^\circ\ 58'$.

26.



Surface density = σ g cm^{-2}

Original sandwich:

Mass = 54σ g

x-coordinate of centre of mass = 3 cm

y-coordinate of centre of mass = 4 cm

Bite:

Mass = $\frac{1}{2}\pi 3^2\sigma = 14.137\ 166\ 94\sigma$ g

Distance of centre of mass from hypotenuse = $\dfrac{4}{3\pi}\times 3 = \dfrac{4}{\pi} = 1.273\ 239\ 545$ cm

x-coordinate of centre of mass = $4.5 - \dfrac{4}{\pi}\sin\theta = 4.5 - \dfrac{16}{5\pi} = 3.481\ 408\ 364$ cm

y-coordinate of centre of mass = $6 - \frac{4}{\pi}\cos\theta = 6 - \frac{12}{5\pi} = 5.236\,056\,273$ cm

Remainder:

Mass = $(54 - 14.137\,166\,94)\sigma = 39.862\,833\,06\sigma$ g

x-coordinate of centre of mass = $\bar{x}$

y-coordinate of centre of mass = $\bar{y}$

Moments:

$39.862\,833\,06\,\bar{x} + 14.137\,166\,94 \times 3.481\,408\,364 = 54 \times 3.$ $\quad \underline{\underline{\bar{x} = 2.829\,270\,780 \text{ cm}}}$

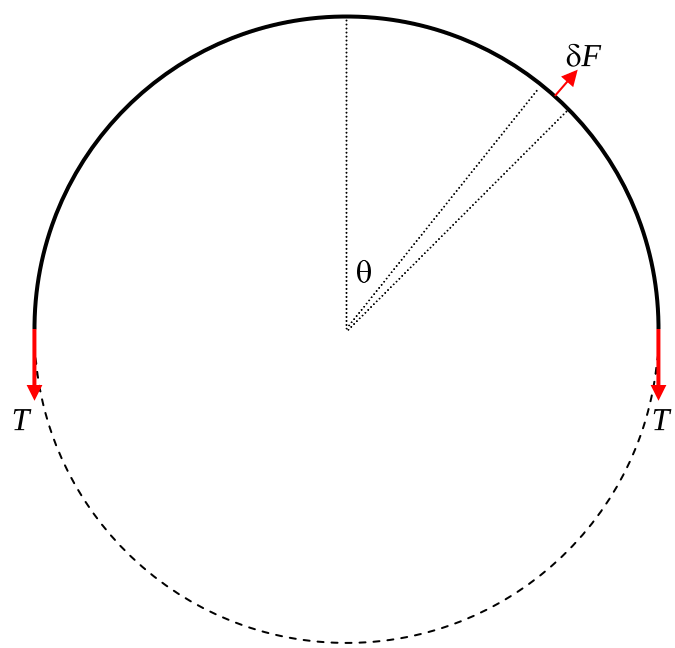
$39.862\,833\,06\,\bar{y} + 14.137\,166\,94 \times 5.236\,056\,273 = 53 \times 4.$ $\quad \underline{\underline{\bar{y} = 3.561\,638\,436 \text{ cm}}}$

This point is very close to the edge of the bite. The centre of the bite is at (4.5, 6), and its radius is 3. Its equation is therefore

$$(x-4.5)^2 + (y-6)^2 = 9, \text{ or } x^2 + y^2 - 9x - 12y + 47.25 = 0.$$

The line $x = 2.829\,270\,780$ cuts the circle where $y^2 - 12y + 29.791\,336\,13 = 0$. The lower of the two points of intersection is at $y = 3.508\,280\,941$ cm. The centre of mass is slightly higher than this and is therefore just inside the bite.

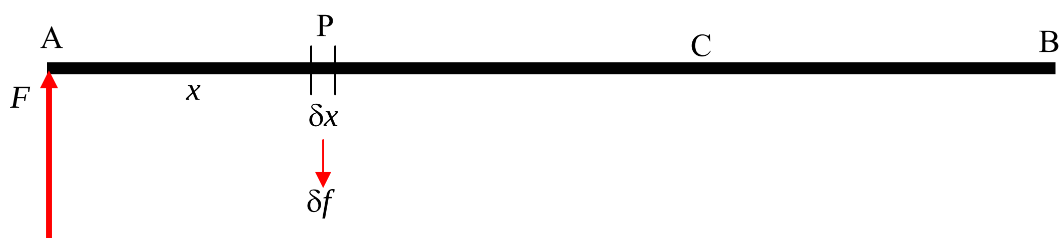
27.



Consider a portion of the band within the angle $\delta\theta$. Its mass is $\frac{m\,\delta\theta}{2\pi}\cdot$ When the band is spinning at angular speed ω and its radius is r, the centrifugal force on that portion is $\delta F = \frac{mr\omega^2\,\delta\theta}{2\pi}\cdot$ (I leave it to the philosophers and the schoolteachers to debate as to whether there "really" is "such thing" as centrifugal force – I want to get this problem done, and I'm referring to a co-rotating frame.) The y-component of this force is $\frac{mr\omega^2\cos\theta\,\delta\theta}{2\pi}\cdot$ Also, the tension in the band when its radius is r is $T = 2\pi k(r-a)$.

Consider the equilibrium of half of the band. The y-component of the centrifugal force on it is $\frac{mr\omega^2}{2\pi}\int_{-\frac{\pi}{2}}^{+\frac{\pi}{2}}\cos\theta\,d\theta = \frac{mr\omega^2}{\pi}$. The opposing force is $2T = 4\pi k(r-a)$. Equating these gives $\underline{\underline{\omega^2 = \frac{4\pi^2 k(r-a)}{mr}}}\cdot$

28. Let the distance AB be l and the distance AC be c. Let the mass of the rod be m.



Consider an elemental portion δx of the rod at P at a distance x from A. Its weight is $\frac{m\,\delta x}{l}\cdot$ When the rod is about to move, it will experience a frictional force $\delta f = \frac{\mu mg\,\delta x}{l}$, which will be in the direction shown if P is to the left of C, and in the opposite direction if P is to the right of C. When the rod is just about to move (but has not yet done so) it is still in equilibrium. Consider the moment about A of the frictional forces on the rod. The clockwise moment of the frictional forces on AC must equal the counterclockwise moment of the frictional forces on CB. Thus

$$\frac{\mu mg}{l}\int_0^c x\,dx = \frac{\mu mg}{l}\int_c^l x\,dx.$$

$\therefore$ $$c = l/\sqrt{2}.$$

The net force on the rod is

$$F - \frac{\mu mg}{l}\int_0^c dx + \frac{\mu mg}{l}\int_c^l dx,$$

and this is zero, and therefore

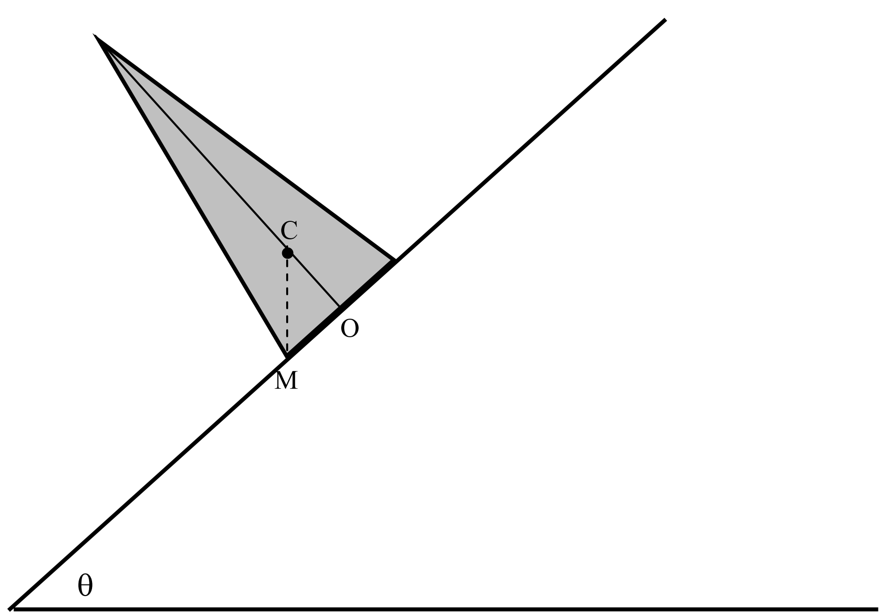
$$F = \frac{\mu mg(2c-l)}{l} = \left(\sqrt{2}-1\right)\mu mg.$$

29. The cone slips when $\tan\theta > \mu$.

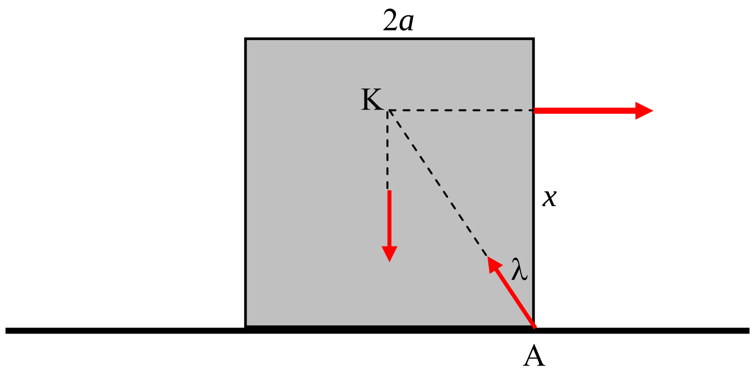
It tips when C (the centre of mass) is to the left of M.

The distance OC is $h/4$. (See Chapter 1, section 1.7). Therefore it tips when $\tan\theta > 4a/h$.

Thus it slips if $\mu < 4a/h$ and it tips if $\mu > 4a/h$.

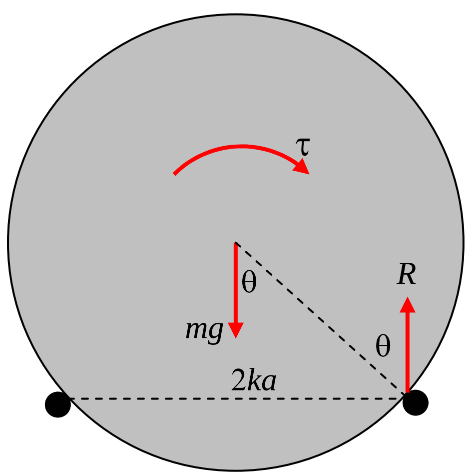


30.



When the block is just about to tip, the reaction of the table on the block acts at A and it is directed towards the point K, because, when three coplanar forces are in equilibrium they must act through a single point. The angle λ is given by $\tan \lambda = a/x$. However, by the usual laws of friction, the block will slip as soon as $\tan \lambda = \mu$. Thus the block will slip if $\mu < a/x$, and it will tip if $\mu > a/x$. Expressed otherwise, it will slip if $x < a/\mu$ and it will tip if $x > a/\mu$. The greatest possible value of x is $2a$; therefore the block will inevitably slip if $\mu < ½$.

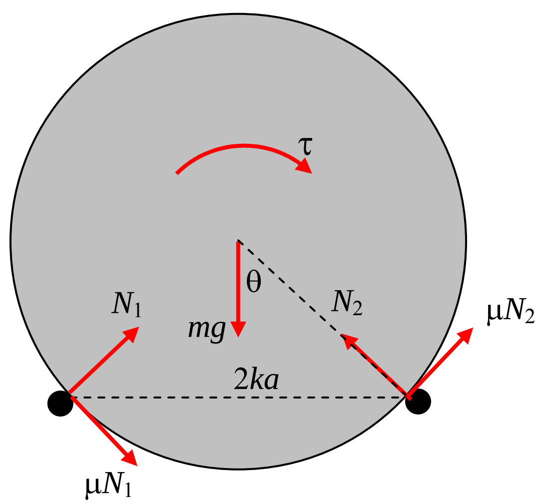
31.



When or if the cylinder is just about to tip, it is about to lose contact with the left hand peg. The only forces on the cylinder are the torque, the weight, and the reaction R of the right hand peg on the cylinder, which must be vertical and equal to mg. But the greatest possible angle that the reaction R can make with the surface of the cylinder is the angle of friction λ given by tan $\lambda = \mu$. From geometry, we see that sin $\theta = k$, or $\tan\theta = k/\sqrt{1-k^2}$. Thus the cylinder will slip before it tips if $\mu < k/\sqrt{1-k^2}$ and it will tip before it slips if $\mu > k/\sqrt{1-k^2}$.

If the cylinder tips (which it will do if $\mu > k/\sqrt{1-k^2}$), the clockwise torque τ at that moment will equal the counterclockwise torque of the couple (R and mg), which is $mgka$. Thus the torque when the cylinder tips is

TIP: $$\tau = mgak. \tag{1}$$



When or if the cylinder is just about to slip, the forces are as shown above, in which I have resolved the reactions of the pegs on the cylinder into a normal reaction (towards the axis of the cylinder) and a frictional force, which, when slipping is about to occur, is equal to μ times the normal reaction. The equilibrium conditions are

$$\mu(N_1 + N_2)\cos\theta + (N_1 - N_2)\sin\theta = 0,$$

$$\mu(N_1 - N_2)\sin\theta - (N_1 + N_2)\cos\theta + mg = 0$$

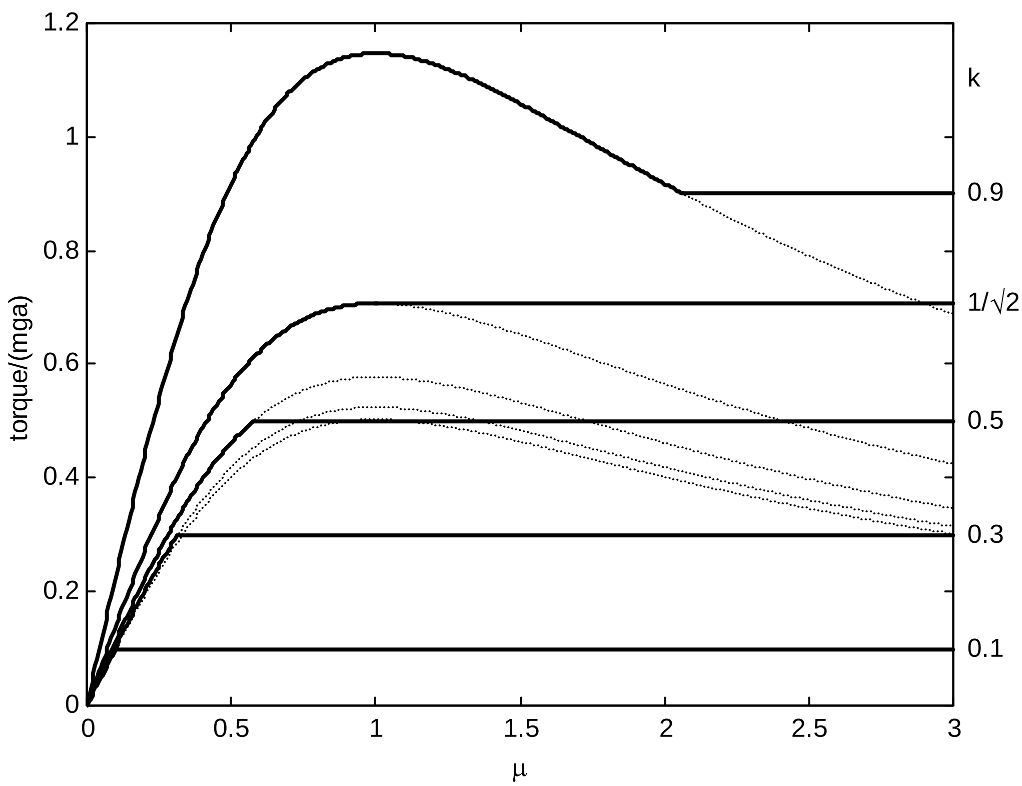
and $$\mu(N_1 + N_2)a = \tau.$$

We can find $N_1 + N_2$ by eliminating $N_1 - N_2$ from the first two equations, and then, writing $\sqrt{1-k^2}$ for cos θ, we find that, when slipping is about to occur, SLIP $$\tau = mga \times \frac{\mu}{1+\mu^2} \times \frac{1}{\sqrt{1-k^2}}. \qquad (2)$$

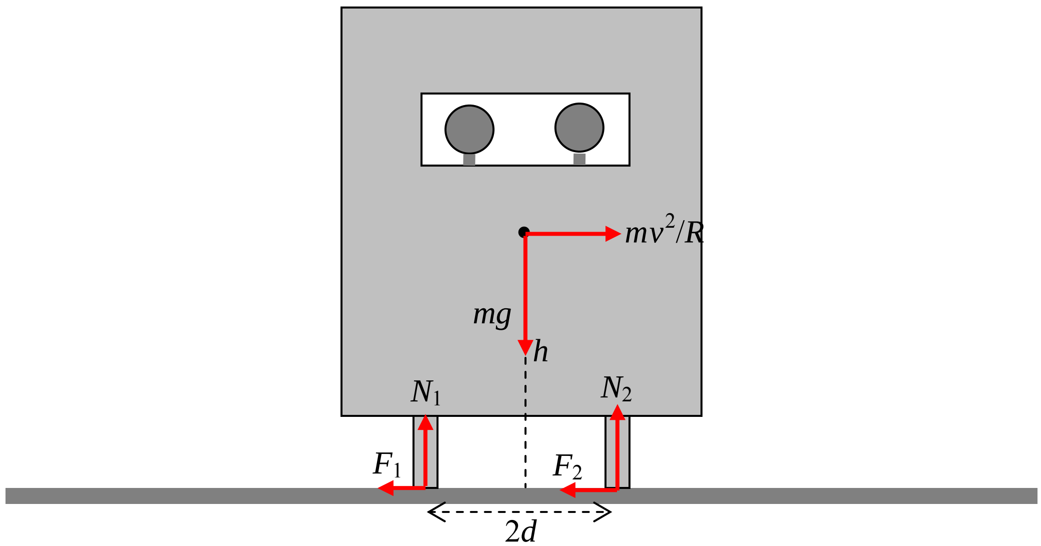
I have drawn below the functions

$$\frac{\tau}{mga} = k \text{ (tip)} \quad \text{and} \quad \frac{\tau}{mga} = \frac{\mu}{1+\mu^2} \times \frac{1}{\sqrt{1-k^2}} \text{ (slip)}$$

for $k = 0.1$, 0.3, 0.5, $1/\sqrt{2}$ and 0.9. The horizontal lines are the tip functions, and the curves are the slip functions. As long as $\mu < k/\sqrt{1-k^2}$ the cylinder will slip. As soon as $\mu > k/\sqrt{1-k^2}$ the cylinder will tip.



32.



We'll leave to the philosophers the question as to whether centrifugal force "really exists", and we'll work in a co-rotating reference frame, so that the car, when referred to that frame, is in static equilibrium under the six forces shown. Clearly, N_1 and $N_2 = mg$ and $F_1 + F_2 = mv^2/R$.

The car slips when $F_1 + F_2 = \mu(N_1 + N_2)$; that is, when $v = \sqrt{\mu g R}$.

The car tips when $mv^2h/R = mgd$; that is, when $v = \sqrt{dgR/h}$.

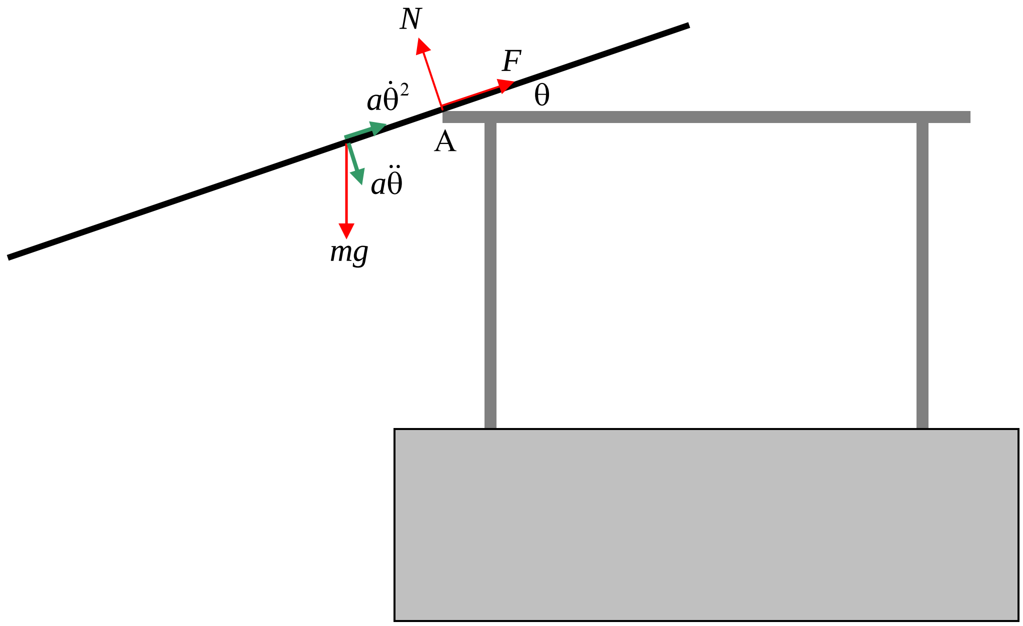
That is, it will slip or tip according as to whether $\mu < d/h$ or $> d/h$.

For example suppose $d = 60$ cm, $h = 80$ cm, $g = 9.8$ m s^{-2}, $R = 30$ m, $\mu = 0.8$.

In that case, $d/h = 0.75$, so it will tip at $v = 14.8$ m s^{-1} = 53.5 km hr^{-1}.

But if it rains, reducing μ to 0.7, it will slip at $v = 14.3$ m s^{-1} = 51.6 km hr^{-1}.

33.



I have drawn in green the radial and transverse components of the acceleration of the centre of mass $a\dot{\theta}^2$ and $a\ddot{\theta}$ respectively. I have drawn in red the weight of the rod and the normal and frictional components of the force of the table on the rod at A, N and F respectively.

The following are the equations of motion:

Normal: $$ma\ddot{\theta} = mg\cos\theta - N. \qquad (1)$$

Lengthwise: $$ma\dot{\theta}^2 = -mg\sin\theta + F. \qquad (2)$$

Rotation: $$k^2\ddot{\theta} = ga\cos\theta. \qquad (3)$$

Here k is the radius of gyration about A, given by

$$k^2 = \tfrac{1}{3}l^2 + a^2. \qquad (4)$$

From equations (1), (3) and (4), we obtain

$$N = mg\cos\theta\left(\frac{l^2}{l^2 + 3a^2}\right). \qquad (5)$$

The space integral (see Chapter 6, section 6.2) of equation (3), with initial condition $\dot{\theta} = 0$ when $\theta = 0$, results in

$$\dot{\theta}^2 = \frac{2ga}{k^2}\sin\theta. \tag{6}$$

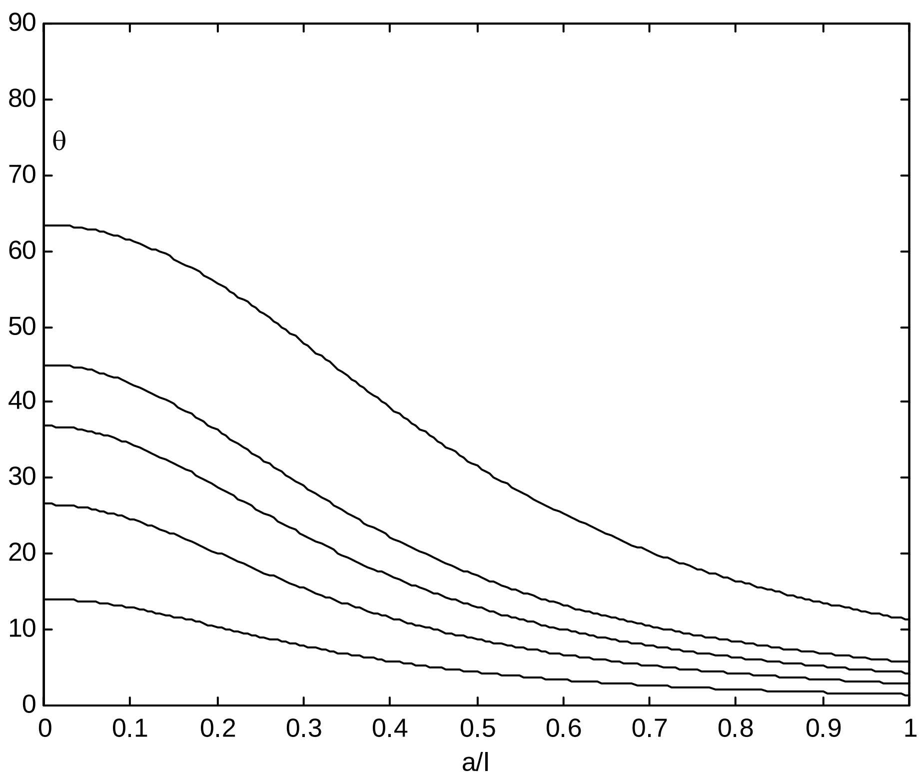
This can also be obtained by equation the loss of potential energy, $mga\sin\theta$, to the gain in kinetic energy, $\frac{1}{2}mk^2\dot{\theta}^2$.

Combining this with equations (2) and (4) leads to

$$F = mg\sin\theta.\left(\frac{l^2 + 9a^2}{l^2 + 3a^2}\right). \tag{7}$$

At the instant of slipping, $F = \mu N$, and hence, from equations (5) and (7) we find

$$\tan\theta = \frac{\mu}{1 + 9(a/l)^2}.$$



34. I derive $v^2 = gx + \frac{g}{l}x^2$ by two different methods – one from energy considerations, the other from angular momentum considerations. First, energy.

If the table top is taken to be the zero level for potential energy, the initial potential energy was $-\frac{1}{2}m.g.\frac{1}{4}l = -\frac{1}{8}mgl$.

When the length of the dangling portion is $\frac{1}{2}l + x$, the potential energy is

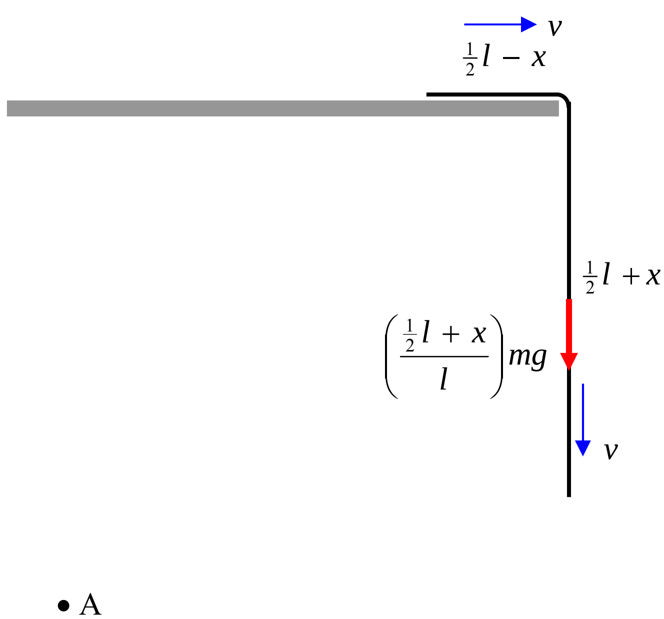
$$-\left(\frac{\frac{1}{2}l + x}{l}\right)m.g.\tfrac{1}{2}(\tfrac{1}{2}l + x) = -\frac{mg}{2l}(\tfrac{1}{2}l + x)^2 = -\tfrac{1}{8}mgl - \tfrac{1}{2}mgx - \frac{mgx^2}{2l}.$$

The loss of potential energy is therefore $\frac{1}{2}mgx + \frac{mgx^2}{2l}$.

This is equal to the gain in kinetic energy $\frac{1}{2}mv^2$, and therefore

$$v^2 = gx + \frac{g}{l}x^2.$$

Another method:



Consider a point A. Anywhere will do, but I have chosen it to be a distance *l* below the level of the table and *l* to the left of the table edge. The moment of momentum (= angular momentum) of the chain about this point is $mlv = ml\dot{x}$, and its rate of change is therefore $ml\ddot{x}$. The torque about A is $\left(\frac{\frac{1}{2}l + x}{l}\right)mgl = (\frac{1}{2}l + x)mg$. These are equal, and so $l\ddot{x} = g(\frac{1}{2}l + x)$. Write $\ddot{x} = v\frac{dv}{dx}$ in the usual way, and integrate (with $v = 0$ when $x = 0$) and the result $v^2 = gx + \frac{g}{l}x^2$ follows.

To find the relation between x and t we can use the energy equation 9.2.9 for conservative systems

$$t = \sqrt{\frac{m}{2}}\int_{x_0}^{x}\frac{dx}{\sqrt{E - V(x)}}.$$

Here $x_0 = 0$ and we have already seen that $E - V(x) = \frac{mg}{2l}x^2 + \frac{mgx}{2}$. Upon integrating this expression, we obtain, after a little algebra and calculus,

$$t = \sqrt{\frac{l}{g}}\ \ln\left(\frac{x + \frac{1}{2}l + \sqrt{x^2 + lx}}{\frac{1}{2}l}\right).$$

The converse of this is the required expression

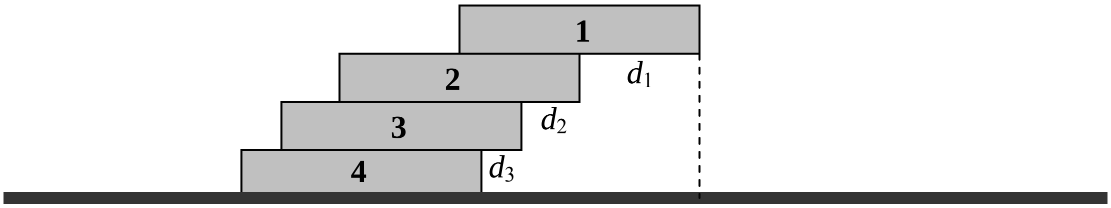
$$x = \frac{l(e^{\sqrt{g/l}\,t} - 1)^2}{4e^{\sqrt{g/l}\,t}}.$$

Differentiation of this with respect to time produces the third required expression:

$$v = \frac{\sqrt{gl}\,(e^{\sqrt{4g/l}\,t} - 1)}{4e^{\sqrt{g/l}\,t}}.$$

You may verify from these last two equations, if you wish, that $v^2 = gx + \frac{g}{l}x^2$.

35. (a)



The maximum overhang of book **1** is $d_1 = w$.

The centre mass of **1**+ **2** is at $3w/2$ from the left hand side (LHS) of **2**, so $d_2 = w/2$.

The distance of the centre of mass of **1**+**2**+**3** is at $5w/2$ from the LHS of **3**, so $d3 = w/3$.

Thus $D = d_1 + d_2 + d_3 = (1 + \frac{1}{2} + \frac{1}{3})w = 1.8\dot{3}w.$

(b) In a similar manner we find that, given $n + 1$ books, the maximum overhang is

$$D = (1 + \tfrac{1}{2} + \tfrac{1}{3} + \ldots \quad \ldots + \tfrac{1}{n})w\,.$$

I don't know if there is a simple expression for the sum to n terms of this harmonic series. Please let me know if you know of one or can find one. Therefore I used a computer to solve

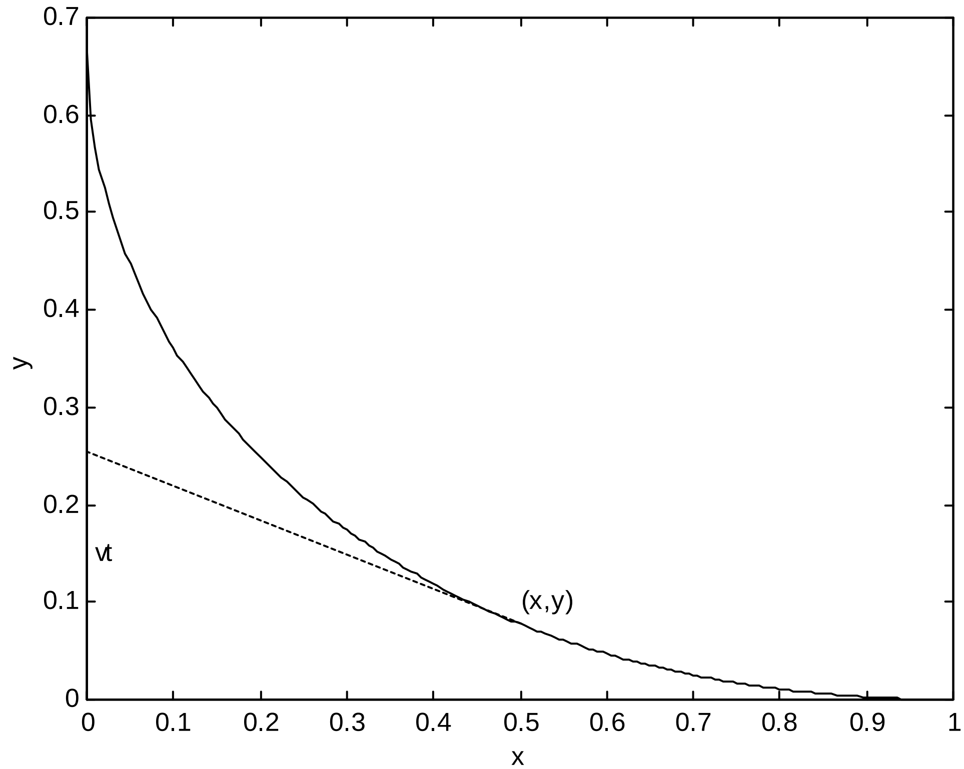
$$1 + \tfrac{1}{2} + \tfrac{1}{3} + \ldots \quad \ldots + \tfrac{1}{n} = 10$$

by brute force. I got $n = 12367$, so you would need 12368 books.

(c) The harmonic series is divergent and has no finite limit, so there is no finite limit to the possible overhang.

You might wish to speculate on any practical limitations on constructing such a pile of books. For example, we have been assuming a uniform gravitational field – but this will no longer be valid once the overhang becomes comparable to the radius of Earth. This will, however, need quite a large number of books.

36. In the solution that follows, a prime (') will be used to denote differentiation with respect to x, and $p = y' = \frac{dy}{dx}\cdot$ I shall also make use of an auxiliary variable $\phi = \sinh^{-1} p$. The initial conditions are $y = 0$, $x = a$, $p = 0$, $\phi = 0$. The final conditions are $x = 0$, $p = -\infty$, $\phi = -\infty$, y to be determined.



At time t, the y-coordinate of the Man is $v\,t$. If $(x\,,\ y)$ are the coordinates of the Dog at that time, the slope of the path taken by the Dog is

$$p \;=\; -\frac{v\,t - y}{x}, \qquad (1)$$

so that

$$v\,t \;=\; y - px. \qquad (2)$$

The speed of the Dog is

$$Av \;=\; -\frac{ds}{dt} \;=\; -\sqrt{1 + p^2}\,\frac{dx}{dt}\,. \qquad (3)$$

[This comes from $ds = \sqrt{1 + (dy/dx)^2}\,dx$. The minus sign is necessary because (dx/dt) is negative, and Av, the speed (not *velocity*!) of the Dog is necessarily positive.]

Now $(dx/dt) \;=\; -1/t'$, so equation (3) can be written

$$Av\,t' = -\sqrt{1+p^2}\,. \qquad (4)$$

If we can eliminate t between equations (2) and (4), we will obtain a relation between the slope p and x, and hence potentially a relation between y and x.

Differentiate equation (2) with respect to x (recalling that $y' = p$):

$$v\,t' = -p'x. \qquad (5)$$

It is now simple to eliminate t' from equations (4) and (5):

$$Ap'x = \sqrt{1+p^2}\,. \qquad (6)$$

On separating the variables and integrating, we obtain

$$A\int\frac{dp}{\sqrt{1+p^2}} = \int\frac{dx}{x}\,. \qquad (7)$$

With initial conditions $p = 0$ when $x = a$, this gives us

$$A\sinh^{-1}p = \ln(x/a), \qquad (8)$$

or

$$x = ae^{A\phi}, \qquad (9)$$

where

$$\phi = \sinh^{-1}p. \qquad (10)$$

Equation (9), with (10), gives us the relation between x and the slope, p. Note that p and hence ϕ are negative, so that equation says that $x < a$.

Our next task will be to find a relation between y and p (or between y and ϕ).

From equation (10) we have

$$dy = \sinh\phi\,dx, \qquad (11)$$

and from equation (9) we have

$$dx = aAe^{A\phi}d\phi. \qquad (12)$$

From these we obtain the differential relation between y and ϕ:

$$dy = aAe^{A\phi}\sinh\phi\,d\phi, \qquad (13)$$

or $$dy = \tfrac{1}{2}aA\left(e^{(A+1)\phi} - e^{(A-1)\phi}\right)d\phi. \tag{14}$$

Integrate this, with initial condition $\phi = 0$ when $y = 0$, to obtain

$$y = \tfrac{1}{2}aA\left(\frac{e^{(A+1)\phi}}{A+1} - \frac{e^{(A-1)\phi}}{A-1} + \frac{2}{A^2-1}\right). \tag{15}$$

Equation (9) and (15) are parametric equations to the path of the Dog, though it is easy to eliminate ϕ and write y explicitly as a function of x:

$$y = \tfrac{1}{2}aA\left(\frac{\left(\frac{x}{a}\right)^{1+1/A}}{A+1} - \frac{\left(\frac{x}{a}\right)^{1-1/A}}{A-1} + \frac{2}{A^2-1}\right). \tag{16}$$

The figure was drawn for $a = 1$, $A = 2$, for which equation (16) reduces to

$$y = \tfrac{1}{3}[x^{1/2}(x-3)+2]. \tag{17}$$

The distance walked by the Man is found by putting $\phi = -\infty$ in equation 15. Thus

$$y = \frac{aA}{A^2-1}, \tag{18}$$

and the time taken is

$$t = \frac{aA}{v(A^2-1)}. \tag{19}$$

37. Let l be the length of the string.

a.

Kinetic energy of the upper mass = $\tfrac{1}{2}(mr^2)\omega^2 + \tfrac{1}{2}m\dot{r}^2$.

Kinetic energy of the lower mass = $\tfrac{1}{2}m\dot{r}^2$.

Potential energy of the lower mass = $= mg(l-r)$.

Total energy of the system = $\tfrac{1}{2}(mr^2)\omega^2 + m\dot{r}^2 - mg(l-r)$. (1)

Initial total energy of the system = $\tfrac{1}{2}(ma^2)\omega_0^2 - mg(l-a)$. (2)

Energy is conserved and therefore, by equating (1) and (2), we obtain

$$\dot{r}^2 = g(a - r) + \tfrac{1}{2}a^2\omega_0^2 - \tfrac{1}{2}r^2\omega^2. \quad (3)$$

Angular momentum is also conserved, and therefore

$$r^2\omega = a^2\omega_0. \quad (4)$$

On elimination of r between equations (3) and (4) we obtain, after some algebra,

$$\frac{\dot{r}^2}{ga} = 1 + \frac{a\omega_0^2}{2g}\left(1 - \frac{\omega}{\omega_0}\right) - \sqrt{\frac{\omega_0}{\omega}} . \quad (5)$$

b. If $a\omega_0^2 = g$ and $\Omega = \omega/\omega_0$, it is trivial to show that

$$\frac{\dot{r}^2}{ga} = \tfrac{3}{2} - \tfrac{1}{2}\Omega - 1/\sqrt{\Omega}. \quad (6)$$

c. Algebra and calculus show that $\tfrac{3}{2} - \tfrac{1}{2}\Omega - 1/\sqrt{\Omega}$ is negative for all positive Ω except for $\Omega = 1$, when it reaches a maximum value of zero.

d. If $a\omega_0^2 = 2g$ and $\Omega = \omega/\omega_0$, it is trivial to show that

$$\frac{\dot{r}^2}{ga} = 2 - \Omega - 1/\sqrt{\Omega}. \quad (7)$$

Algebra and calculus show that $2 - \Omega - 1/\sqrt{\Omega}$ reaches a maximum value for $\Omega = \omega/\omega_0 = 1/2^{2/3} = 0.629\,961$, at which time $\dot{r}^2/(ga) = 0.110\,118$. That is, when $\dot{r} = 0.331\,841\sqrt{ga}$. Equation (4) (conservation of angular momentum) shows that $r = a/\sqrt{\Omega} = \sqrt[3]{2}a = 1.259\,921a$.

Solution of $2 - \Omega - 1/\sqrt{\Omega} = 0$ shows that the speed is zero when $\Omega = 1$ (the initial condition) and when $\Omega = \omega/\omega_0 = 0.381\,966$ (the equilibrium value). Equation (4) (conservation of angular momentum) shows that $r = a/\sqrt{\Omega} = 1.618\,034a$.

e. If $a\omega_0^2 = \tfrac{1}{2}g$ and $\Omega = \omega/\omega_0$, it is trivial to show that

$$\frac{\dot{r}^2}{ga} = \tfrac{5}{4} - \tfrac{1}{4}\Omega - 1/\sqrt{\Omega}. \qquad (8)$$

Algebra and calculus show that $\tfrac{5}{4} - \tfrac{1}{4}\Omega - 1/\sqrt{\Omega}$ reaches a maximum value for $\Omega = \omega/\omega_0 = 2^{2/3} = 1.587\,401$, at which time $\dot{r}^2/(ga) = 0.059\,449$. That is, when $\dot{r} = -0.243\,822\sqrt{ga}$. Equation (4) (conservation of angular momentum) shows that $r = a/\sqrt{\Omega} = a/\sqrt[3]{2} = 0.793\,701a$.

Solution of $\tfrac{5}{4} - \tfrac{1}{4}\Omega - 1/\sqrt{\Omega} = 0$ shows that the speed is zero when $\Omega = 1$ (the initial condition) and when $\Omega = \omega/\omega_0 = 2.438\,447$ (the equilibrium value). Equation (4) (conservation of angular momentum) shows that $r = a/\sqrt{\Omega} = 0.640\,388a$.

How much further can we go with this question? By elimination of r between equations (3) and (4) we obtained a relation between $\dot{r}$ and ω. By elimination of ω between equations (3) and (4) we can get a relation between $\dot{r}$ and r. It will be of the form

$$\dot{r} = \sqrt{A - gr - B/r^2}, \qquad (9)$$

where $A = ga + \tfrac{1}{2}a^2\omega_0^2$ and $B = a^4\omega_0^4$. If you can integrate this, you then get a relation between r and t. I haven't given much though as to whether you can get integrate equation (9) analytically (if anyone manages it, please let me know), but at least a numerical integration will certainly be possible.

In another variation of the question, you can start with an equilibrium situation in which $a\omega_0^2 = g$, and then add an extra mass m (or M, if you want to make it more general) and then follow the motion from there. I leave that to you.